MATH 113: DISCRETE STRUCTURES
MONDAY WEEK 2 HANDOUT

The *floor* function $\lfloor\ \rfloor : \mathbb{R} \to \mathbb{R}$ sends $x \in \mathbb{R}$ to the greatest integer less than or equal to x. For instance, $\lfloor 4.5 \rfloor = 4$, $\lfloor 17 \rfloor = 17$, and $\lfloor -\pi \rfloor = -4$.

Problem 1. Draw a graph of $\lfloor\ \rfloor$ and check that it is a function. What is the image of the floor function? Is it injective or surjective?

Problem 2. Define $f : \mathbb{N} \to \mathbb{Z}$ by

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ \frac{1-n}{2} & \text{if } n \text{ is odd.} \end{cases}$$

Show that f is a bijection.

Problem 3. Suppose A and B are finite sets and $f : A \to B$ is injective. What can we say about $|A|$ and $|B|$? What if f is surjective?

Problem 4. Let $F(A, B)$ denote the set of functions with domain A and codomain B. If $|A|, |B| < \infty$, what is $|F(A, B)|$? (In other words, how many functions are there with domain A and codomain B?)

Suppose A and B are sets and $f : A \to B$ is a function. If $A' \subseteq A$, then the *image* of A' in B is defined as

$$f(A') := \{f(a) \mid a \in A'\}.$$

Note that $f(A) = \mathrm{im}(f)$. If $B' \subseteq B$, then the *preimage* of B' in A is defined as

$$f^{-1}(B') := \{a \in A \mid f(a) \in B'\}.$$

In other words, $f^{-1}(C)$ consists of everything in A pushed into C by f.

Problem 5. Determine $f(\varnothing)$ and $f^{-1}(\varnothing)$. More generally, when is $f^{-1}(B') = \varnothing$?

Problem 6. For $A_1, A_2 \subseteq A$, $B_1, B_2 \subseteq B$, and $f : A \to B$, prove that

$$\begin{aligned} f(A_1 \cup A_2) &= f(A_1) \cup f(A_2), \\ f(A_1 \cap A_2) &\subseteq f(A_1) \cap f(A_2), \\ f^{-1}(B_1 \cup B_2) &= f^{-1}(B_1) \cup f^{-1}(B_2), \text{ and} \\ f^{-1}(B_1 \cap B_2) &= f^{-1}(B_1) \cap f^{-1}(B_2). \end{aligned}$$

Find an example to show that equality does not necessarily hold in the second line.