MATH 113: DISCRETE STRUCTURES
FRIDAY WEEK 2 HANDOUT

For $n \in \mathbb{N}$, let $\underline{n} = \{1, 2, \ldots, n\}$. In particular $\underline{1} = \{1\}$, $\underline{2} = \{1, 2\}$, $\underline{3} = \{1, 2, 3\}$, etc. Note that $\underline{0} = \varnothing$ by convention.

Problem 1. There are k^n length n strings where each entry in the string comes from a set with k elements. Earlier, you proved that there are k^n functions with domain $\underline{n}$ and codomain $\underline{k}$. Is this a coincidence? Explain.

We take the viewpoint that a permutation is a bijection from a set to itself. This can also be though of as a reordering of the set. If $\pi : \underline{n} \to \underline{n}$ is a bijection, it reorders $\underline{n}$ from $1, 2, \ldots, n$ to $\pi(1), \pi(2), \ldots, \pi(n)$. This also gives us the analogy string:function::reordering:permutation. In particular, we may view permutations of $\underline{n}$ as length n strings with entries in $\underline{n}$ in which no 'letters' are repeated.

Problem 2. Why does this prove that $n! \leq n^n$? What do you think $n!/n^n$ approaches as n goes to ∞?

Define the *sign* of a permutation $\pi : \underline{n} \to \underline{n}$ by the formula

$$\operatorname{sgn}(\pi) = \prod_{1 \leq i < j \leq n} \frac{\sigma(j) - \sigma(i)}{j - i}.$$

Problem 3. Write out the formula for $\operatorname{sgn}(\pi)$ when $n = 3$. Why is it the case that $\operatorname{sgn}(\pi) = \pm 1$ in this case? Show that $\operatorname{sgn}(\pi) \in \{\pm 1\}$ for all n.

Note that we can compose permutations to get new permutations when thinking of them as bijective functions $\underline{n} \to \underline{n}$.

Problem 4. For π, τ permutations of $\underline{n}$, prove that $\operatorname{sgn}(\pi \circ \tau) = \operatorname{sgn}(\pi) \cdot \operatorname{sgn}(\tau)$.