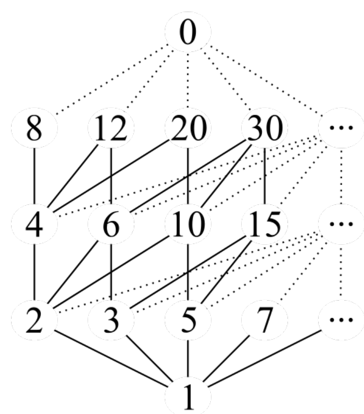
MATH 113: DISCRETE STRUCTURES
MONDAY WEEK 11 HANDOUT

For integers a, b, we say that a *divides* b when an integer m exists such that $b = am$; in this case we also say that b *is a multiple of* a and that a *is a divisor of* b.

Question 1. When does $1 \mid b$? $-1 \mid b$? $a \mid 0$? $a \mid a$?

Problem 2. Suppose that $a \mid b$ and $b \mid c$. Prove that $a \mid c$.

This produces a *partial order* on $\mathbb{N}$, visualized in the following diagram.



Question 3. Where should you put 9 in the diagram?

Problem 4. Prove that if $a \mid b$ and $a \mid c$, then $a \mid b + c$ and $a \mid b - c$.

A natural number $p > 1$ is *prime* if its only positive divisors are 1 and p. The fundamental theorem of arithmetic says that every positive integer is a product of primes, and that this factorization is unique up to reordering of the factors. For instance, $6 = 2 \cdot 3$, $1728 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3 = 2^6 \cdot 3^3$ and $825 = 3 \cdot 5 \cdot 5 \cdot 11 = 3 \cdot 5^2 \cdot 11$. This probably seems like old hat, but not every number system has unique factorization! For instance, $\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\}$ supports addition and multiplication, but

$$6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5}).$$

Number theorists are quite interested in objects like $\mathbb{Z}[\sqrt{-5}]$, but we will limit our study to $\mathbb{Z}$ where the fundamental theorem of arithmetic holds.

Question 5. Where should the prime numbers go in the divisibility diagram?

Problem 6. Prove that a positive integer n is prime if and only if n is not divisible by any prime p with $1 < p \leq \sqrt{n}$.

Problem 7. Suppose that a positive integer n has prime factorization $n = p_1 p_2 \cdots p_k$. How many distinct positive integers are divisors of n?

Problem 8. The book's proof does a fine job of guaranteeing that prime factorizations of integers are unique, but it elides the proof that prime factorization *exist*. Give an inductive proof that every positive integer has a prime factorization.