

**MATH 113: DISCRETE STRUCTURES**  
**WEDNESDAY WEEK 10 HANDOUT**

*Problem 1.* The digits 1, 2, 3, 4 are randomly arranged into two two-digit numbers  $\overline{AB}$  and  $\overline{CD}$ . In this problem you will ultimately determine the expected value of  $\overline{AB} \cdot \overline{CD}$ .

- (a) If two of the digits 1, 2, 3, 4 are randomly selected (without replacement), what is their expected product?
- (b) Write  $\overline{AB}$  as a linear combination of the digits  $A$  and  $B$ . Similarly express  $\overline{CD}$  in terms of  $C$  and  $D$ .
- (c) Finally, use linearity of expectation and your answer to (a) to determine  $E(\overline{AB} \cdot \overline{CD})$ .

*Problem 2* (The coupon collector problem). Safeway is running a promotion in which they have produced  $n$  coupons and you randomly receive a coupon each time you check out. You passionately hope to one day collect all  $n$  coupons. What is the expected number of times  $T$  you'll have to check out at the store in order to collect all  $n$ ? There's a very clever way to solve this problem with linearity of expectation!

- (a) Label the coupons  $C_1, C_2, \dots, C_n$ . If  $n = 4$ , a successful collection of all 4 coupons might look like  $C_2 C_2 C_4 C_2 C_1 C_3$ . Break the sequence into segments where a segment ends when you receive a new coupon. In the example sequence, the segments are  $C_2, C_2 C_4, C_2 C_1, C_3$ . Because it will make our lives easier and Kyle is a benevolent problem-writer, consider these the 0-th, 1-st,  $\dots$ , 3-rd segments (as opposed to 1-st through 4-th). Let  $X_k$  be the length of the  $k$ -th segment, and note that  $k$  ranges from 0 through  $n - 1$ . In the example,  $X_0 = 1, X_1 = 2, X_2 = 2$ , and  $X_3 = 1$ . Express  $T$ , the total number of checkouts needed to collect all coupons, as a linear combination of the  $X_k$ .
- (b) Compute  $p_k$ , the probability that you will collect a new coupon given that you have already collected  $k$  of them. After studying the geometric distribution in Lecture 5, we will learn that  $E(X_k) = 1/p_k$ . Compute this value.
- (c) Use your answers to (a) and (b) to determine  $E(T)$ .
- (d) Can you say anything about the asymptotic behavior of  $E(T)$ ?