MATH 113: DISCRETE STRUCTURES
EXAM 2 REVIEW

For the second exam, you are responsible for the content covered in class, our reading assignments, and video lectures through Friday, March 29. The emphasis of the exam will strongly skew towards content covered after the first exam. Topics include: derangements, recurrence, the Fibonacci numbers, graph theory, and discrete probability (distributions, independence, Bayes' theorem, total probability, random variables, expectation and linearity thereof, binomial and geometric distributions).

The exam is a take-home 120-minute exam. You must take the exam on your own in a distraction-free environment. The time you spend on the exam must be continuous, not broken into multiple pieces. You may reference one two-sided sheet of notes (US letter size or A4 paper). No other resources (including books, other notes, calculators, phones, computers, or consultation with other individuals) are permitted.

The following problems are for practice and are intended to help you review for the exam. They are not to be turned in and will not be graded, though I am happy to provide feedback if you have questions about a solution.

One of these problems will appear on the exam, but most of these problems are *harder* than exam problems.

Problem 1. Find a closed formula for a_n where $a_0 = 0$, $a_1 = 2$, and $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$. What if $a_0 = 7$ and $a_1 = 9$? Prove both your assertions.

Problem 2. Recall that two graphs are *isomorphic* if there is a bijection between their vertex sets which induces a bijection between edge sets.

(a) Prove that isomorphic graphs have the same number of vertices, the same number of edges, and the same list of vertex degrees.
(b) After finishing part (a), an overeager student asserts that graphs with the same number of vertices, the same number of edges, and the same list of vertex degrees are always isomorphic. Prove that this student is wrong by exhibiting non-isomorphic graphs that have these properties.

Problem 3. Recall that $K_{p,q}$ is the complete bipartite graph on $p + q$ vertices. For which p, q does $K_{p,q}$ have an Eulerian circuit?

Problem 4. Suppose A and B are two events with $P(A) = 0.5$, $P(A \cup B) = 0.8$.

(a) For what values of $P(B)$ would A and B be mutually exclusive?
(b) For what values of $P(B)$ would A and B be independent?

Problem 5. A digital communications system consists of a transmitter and a receiver. During each short transmission interval the transmitter sends a signal to be interpreted as a 0 or a 1. At the end of each interval, the receiver makes its best guess at what was transmitted. For $i = 0, 1$, let T_i be the event that the transmitter sends i and let R_i be the event that the receiver concludes that i was sent. Assume that $P(R_0|T_0) = 0.99$, $P(R_1|T_1) = 0.98$, and $P(T_1) = 0.5$.

(a) What is the probability of a communication error given R_1?
(b) What is the overall probability of a communication error?

Problem 6. A building has 10 floors above the ground floor and no basement. If 12 people get into the elevator at the ground floor, and each chooses a non-ground floor at random to get out, independently of the others, at how many floors do you expect the elevator to make a stop?

Problem 7 (Indicator variables and inclusion-exclusion). Let I_A be the indicator variable for an event $A \subseteq S$. Recall that $I_A(s) = 1$ if $s \in A$ and otherwise $I_A(s) = 0$; further recall that $E(I_A) = P(A)$.

(a) Prove that $I_{A^c} = 1 - I_A$.
(b) Prove that $I_{A \cap B} = I_A I_B$.
(c) Prove that for any $A_1, A_2, \ldots, A_n \subseteq S$,

$$I_{A_1 \cup A_2 \cup \cdots \cup A_n} = 1 - (1 - I_{A_1})(1 - I_{A_2}) \cdots (1 - I_{A_n}).$$

(d) Expand the product in the formula from (c) and use linearity of expectation to derive the inclusion-exclusion formula.

Problem 8. Suppose you pick people at random and ask them their birthday. Let X be the number of people you have to question until you find a person who was born in November. What is $E(X)$ (assuming birthdays are distributed uniformly randomly, and ignoring the existence of leap days)?

Problem 9. In this problem, we will construct a random graph on vertex set $\underline{n}$ by including each potential edge $\{i, j\}$ with probability p ($0 \leq p \leq 1$). We define a *triangle* in a graph to be a triple of vertices $\{i, j, k\}$ such that each edge $\{i, j\}$, $\{j, k\}$, $\{k, i\}$ is in the graph. Given the above setup, what is the expected number of triangles in a random graph (with edge probability p)?