MATH 113: DISCRETE STRUCTURES
WEDNESDAY WEEK 9 HANDOUT

Problem 1. The digits 1, 2, 3, 4 are randomly arranged into two two-digit numbers $\overline{AB}$ and $\overline{CD}$. In this problem you will ultimately determine the expected value of $\overline{AB} \cdot \overline{CD}$.

(a) If two of the digits 1, 2, 3, 4 are randomly selected (without replacement), what is their expected product?

(b) Write $\overline{AB}$ as a linear combination of the digits A and B. Similarly express $\overline{CD}$ in terms of C and D.

(c) Finally, use linearity of expectation and your answer to (a) to determine $E(\overline{AB} \cdot \overline{CD})$.

Problem 2 (The coupon collector problem). Safeway is running a promotion in which they have produced n coupons and you randomly receive a coupon each time you check out. You passionately hope to one day collect all n coupons. What is the expected number of times T you'll have to check out at the store in order to collect all n? There's a very clever way to solve this problem with linearity of expectation!

(a) Label the coupons C_1, C_2, $\ldots$, C_n. If $n = 4$, a successful collection of all 4 coupons might look like $C_2\ C_2\ C_4\ C_2\ C_1\ C_3$. Break the sequence into segments where a segment ends when you receive a new coupon. In the example sequence, the segments are C_2, $C_2\ C_4$, $C_2\ C_1$, C_3. Because it will make our lives easier and Kyle is a benevolent problem-writer, consider these the 0-th, 1-st, …, 3-rd segments (as opposed to 1-st through 4-th). Let X_k be the length of the k-th segment, and note that k ranges from 0 through $n-1$. In the example, $X_0 = 1$, $X_1 = 2$, $X_2 = 2$, and $X_3 = 1$. Express T, the total number of checkouts needed to collect all coupons, as a linear combination of the X_k.

(b) Compute p_k, the probability that you will collect a new coupon given that you have already collected k of them. After studying the geometric distribution in Lecture 5, we will learn that $E(X_k) = 1/p_k$. Compute this value.

(c) Use your answers to (a) and (b) to determine $E(T)$.

(d) Can you say anything about the asymptotic behavior of $E(T)$?