MATH 113: DISCRETE STRUCTURES
FRIDAY WEEK 6 HANDOUT

Problem 1. In this problem we will determine the number of regions in the plane created by a system of n mutually overlapping circles in general position. By *mutually overlapping*, we mean that each pair of circles intersects in two distinct points. By *general position*, we mean that there are no three circles through a common point. Let a_n be the number of regions created by such a system.

(a) Draw some pictures to determine a_0, a_1, a_2, and a_3.
(b) Do you have a conjecture regarding the value of a_n? Check it by drawing a picture to determine a_4.
(c) Take a system of $n-1$ circles (creating a_{n-1} regions) then add an n-th circle which is mutually overlapping and in general position. How many times does this circle intersect circles in the system of $n-1$ circles? How many arcs on the new circle are created by these intersections?
(d) Use your above analysis to determine a recurrence relation which a_n satisfies. (For which n does the recurrence relation hold?)
(e) Use your recurrence relation to find a closed formula (only in terms of n) for a_n (at least for n sufficiently large).
(bonus) Can you find a direct (as opposed to recurrence-based) argument for your formula in (e)?

Problem 2. An anxious ant wanders through a 3×3 grid of the form

1	2	3
4	5	6
7	8	9

and only passes between cells via edges (as opposed to corners). We would like to count the number p_n of length n paths the ant can take where there is no constraint on where the ant starts or ends the path. (A "step" in the path is when the ant changes cells, despite the fact that this takes the ant many many steps. We do not permit the "stay put" step.) A direct recurrence relation on p_n is difficult to come by. (If the $(n-1)$-th step is to cell 1, then the ant can only travel to 2 or 4, but if the $(n-1)$-th step is to cell 5, the ant can travel to 2, 4, 6, or 8.) Instead, we seek multiple recurrence relations (and some good luck).

(a) Let a_n denote the number of length n paths ending in 1, let b_n denote the number of length n paths ending in 2, and let c_n denote the number of length n paths ending in 5. What is the relationship between p_n and these three sequences. (Use symmetry!)
(b) Determine a *system of recurrence relations* for the sequences a_n, b_n, c_n. (This is like a recurrence relation, but each sequence may depend on previous terms of the other sequences.)
(c) Use algebra to find a recurrence relation for b_n (only in terms of previous terms from the same sequence).
(d) Put everything together to get a recurrence relation for p_n.
(e) Compute p_0, p_1, p_2, p_3, p_4, and p_5. Why is the ant anxious?