MATH 113: DISCRETE STRUCTURES
WEDNESDAY WEEK 13 HANDOUT

Problem 1. Suppose G is planar with k connected components. Prove that

$$v - e + f = k + 1.$$

In the reading, you learned that a connected planar graph on v vertices has at most $3v - 6$ edges. By the Euler formula, we thus know that in a planar graph

$$v - e + f = 2$$
$$e \geq 3v - 6.$$

The proof that $e \geq 3v - 6$ proceeds by arguing that every face has at least 3 edges. We can improve this bound if we know something about the *girth* of the graph.

Definition 2. The *girth* g of a graph G is the length of the smallest cycle in G.

Problem 3. Prove that in a planar graph of girth g, $f \leq 2e/g$. Combine this with Euler's formula to prove that

$$e \leq \frac{g}{g-2}(v-2).$$

Note that $g/(g-2)$ is a decreasing function of g for $g > 2$, and in fact $g \geq 3$ for all graphs. We conclude that this inequality is least stringent when $g = 3$, in which case we recover $e \leq 3v - 6$. The encompassing moral is that planar graphs have relatively few edges!

Problem 4. Prove that every planar graph contains a vertex of degree ≤ 5. (*Hint*: Proceed by contradiction and use vertex degrees to count edges. You should ultimately contradict the known inequality $e \leq 3v - 6$.)

Problem 5. Is there a planar graph in which every vertex has degree 5? (*Hint*: Get Platonic.)

The text proves that K_5 is not planar, and you will prove in your homework that $K_{3,3}$ is not planar. Any student of graph theory should be aware of the following theorem (which we won't have time to prove):

Theorem 6 (Kuratowski)**.** *Every non-planar graph contains a subgraph which is a subdivision*[1] *of K_5 or $K_{3,3}$.*

A (good) *coloring* of a graph $G = (V, E)$ by color set C is a function $c : C \to V$ that assigns a color to each vertex such that $c(v) = c(w)$ implies $\{v, w\} \notin E$. In other words, vertices connected by an edge don't have the same color. A graph is called *n-colorable* if it has a good coloring with $\leq n$ colors.

Problem 7. Prove (by induction?) that every planar graph is 6-colorable.

The famous Four Color Theorem says that every planar graph is in fact 4-colorable, but the proof is not easy!

[1] A graph H is a subdivision of G if it can be obtained by turning some of G's edges into paths.