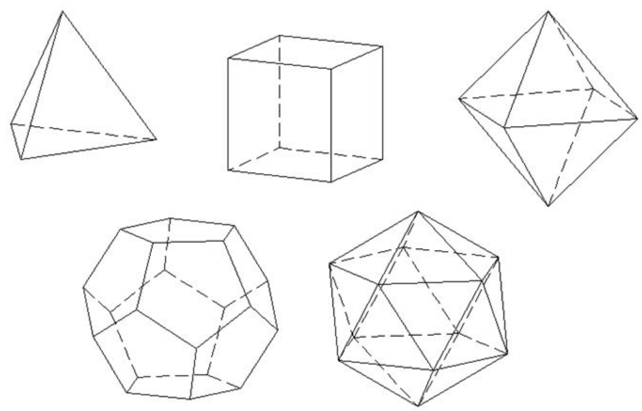
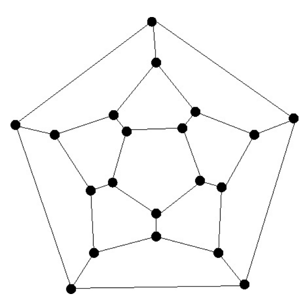
MATH 113: DISCRETE STRUCTURES
FRIDAY WEEK 13 HANDOUT

A *polyhedron* is a subset of $\mathbb{R}^3$ built from flat polygonal faces joined along their edges and vertices. A polyhedron P is *convex* if for any two points $x, y \in P$, the line segment joining x and y remains in P. A convex polyhedron is *regular* if all its faces are regular polygons and the same number of faces meet at each vertex. A convex regular polyhedron is called a *Platonic solid*.

Theorem 1. *There are exactly five Platonic solids: the regular tetrahedron, cube, octahedron, dodecahedron, and icosahedron.*



We can use Euler's formula for planar graphs to prove this theorem! But first, we need a way to extract a planar graph from a convex polyhedron. The idea is to place a light just above one of the faces of the polyhedron and record the shadow of the polyhedron. For instance, here is the graph of the dodecahedron:



Since this graph is planar, we know that if a convex polyhedron has v vertices, e edges, and f faces, then $v - e + f = 2$.

Outline Proof of Theorem 1. With your group, fill in the details in this outline in order to prove the theorem. Throughout, suppose that P is a Platonic solid with v vertices, e edges, and f faces.

(1) Suppose that each face of P is a regular triangle. We aim to show that P must be the regular tetrahedron, octahedron, or icosahedron.
   (a) Show that $e = 3f/2$.
   (b) Use Euler's formula to conclude that $v = 2 + f/2$.
   (c) Let d denote the degree of each vertex. Show that $e = dv/2$.

(d) Combine these observations to conclude that

$$d = \frac{6f}{4+f}.$$

(e) Observe that $6f/(4+f)$ is an increasing function with horizontal asymptote $d = 6$. Conclude that $d = 3$, 4, or 5, corresponding to the tetrahedron, octahedron, or icosahedron, respectively.

(2) Suppose that each face of P is a square. We aim to show that P must be a cube.

(a) Show that $e = 2f$.

(b) Show that $v = 2 + f$.

(c) Show that $e = d(2+f)/2$ where d is the degree of each vertex.

(d) Combine these observations to conclude that

$$d = \frac{8f}{4+2f}.$$

(e) The function $8f/(4+2f)$ is increasing with horizontal asymptote $d = 4$. Conclude that $d = 3$, and that this corresponds to a cube.

(3) Suppose that each face of P is a regular pentagon. We aim to show that P must be the regular dodecahedron.

(a) Mimic the steps from the previous two parts to prove that

$$d = \frac{10f}{4+3f}.$$

(b) This function is increasing with horizontal asymptote $d = 10/3$. Conclude that $d = 3$, and that this corresponds to a dodecahedron.

(4) Suppose that each face of P is a regular n-gon for $n \geq 6$. We aim to show that no such P exists.

(a) As above, show that

$$d = \frac{2nf}{4+(n-2)f}.$$

(b) This function is increasing with horizontal asymptote $d = 2n/(n-2) \leq 3$. Hence $d < 3$, which is absurd, so no such P exists.

□