MATH 113: DISCRETE STRUCTURES
MONDAY WEEK 11 HANDOUT

Problem 1. As an intrepid wagon wheel painter living in the Olde West, you strive to bring the highest quality, most engaging, non-monochromatic spoke paintings to your customers. You offer wagon wheels with p spokes, where p is a prime integer, painted in up to a colors, where $1 \leq a \leq p - 1$.

(a) As part of your preparation for painting, you have nailed a wagon wheel to the wall so that it can't rotate. In how many ways can you paint its spokes, assuming that each spoke gets a single color but at least two of the spokes are different colors?

(b) When you take the wheel off of the wall and fix it to an axle, you remember that it will rotate, and that your demanding customers will not accept rotated spoke paintings as genuinely different. As you turn this particular wheel around, you notice something remarkable: all of the rotations by multiples of $2\pi/p$ result in distinct colorings in the wheel-nailed-to-wall sense of unique, despite the fact that there are multiple spokes of the same color (since $a < p$). Is this a special property of your particular spoke painting, or is it true of all possible non-monochromatic paintings with a colors?

(c) Use your work in (b) to determine the total number wagon wheel paintings which your customers will accept as genuinely different. What can you deduce from the fact that this number is an integer?

Problem 2. How many 6-spoke wheels can you paint non-monochromatically with up to a colors for $a = 2, 3, 4, 5$?