MATH 113: DISCRETE STRUCTURES
WEDNESDAY WEEK 10 HANDOUT

The key takeaways from §6.4 are that there are infinitely many prime numbers, and that the prime counting function $\pi(n) = |\{p \in \mathbb{N} \text{ prime} \mid p \leq n\}|$ grows like $n/\log n$. (Here we are using $\log$ for the natural logarithm function.) The first of these results is generally attributed to Euclid, c. 300B.C.E. Let's look at another proof due to Filip Saidak from 2005. In order to get it off the ground, prove the following result.

Problem 1. Let n be a positive integer. Prove that n and $n+1$ share no common divisors greater than 1.

Proof that there are infinitely many prime numbers. Let $n > 1$ be a positive integer. As we have just proven, n and $n+1$ share no common divisors greater than 1. Hence the number $N_2 = n(n+1)$ must have at least two distinct prime factors. Similarly, N_2 and $N_2 + 1$ share no common divisors greater than 1, and thus $N_3 = N_2(N_2+1)$ must have at least 3 distinct prime factors. We recursively define $N_k = N_{k-1}(N_{k-1}+1)$ for $k > 2$ and observe inductively that N_k has at least k distinct prime factors. □

Note that N_k has at least k distinct prime factors, each of which is necessarily smaller than N_k. It follows that $\pi(N_k) \geq k$.

Question 2. Compute N_k for $2 \leq k \leq 5$. Is this a very effective bound on the prime counting function?

The vaunted Prime Number Theorem (PNT) says that

$$\pi(n) \sim \frac{n}{\log n},$$

which means that

$$\lim_{n\to\infty} \frac{\pi(n)}{n/\log n} = \lim_{n\to\infty} \frac{\pi(n)\log n}{n} = 1.$$

The proof is very difficult and beyond the scope of this course, but we will still happily use the result.

Problem 3. Show that $\lim_{n\to\infty} \pi(n)/n = 0$ and use this to show that for any $a \in \mathbb{R}$,

$$\pi(n) \sim \frac{n}{\log(n) - a}.$$

It turns out that $a = 1$ gives the best approximation to $\pi(n)$. In the below plot, the curve on top is the graph of $n/(\log(n) - 1$, the middle curve is the graph of $\pi(n)$, and the bottom curve is the graph of $n/\log n$.