Trust the Cones: Computing Diagonal Cartier Algebras of Affine Toric Varieties

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Abstract

In this thesis, we describe a geometric method for computing diagonal Cartier algebras of toric rings arising from two-dimensional cones. We introduce the geometric tools necessary to define toric rings and the positive characteristic algebra that goes into the construction of Cartier algebras. We then present Smolkin's work to re-characterize diagonal Cartier algebras geometrically. In Chapter 3 we present our results and list potential avenues for future research.

Introduction

Given a parallelogram P in $\mathbb{R}^2$, there are a variety of interesting geometric questions one can ask about P. Of these, one question that will often crop up is whether or not P can cover $\mathbb{R}^2$ via integer translates. Since this question is not so difficult to answer, the main problem we can consider will be a generalization of this question.

If we have a parallelogram P with upper right-hand vertex x and a vector $v \in \mathbb{R}^2$, we can then let P_v be the parallelogram whose upper right-hand vertex is $v + x$ and whose sides remain parallel to the sides of P. Below, in Figure 1 we have on the left a parallelogram P outlined in blue shown and on the right we have P_v outlined in green where $v = (-\frac{1}{5}, -\frac{1}{5})$. From here, we can then ask the following question: for a parallelogram P what is the vectors $v \in \mathbb{R}^2$ such that a given parallelogram P_v covers $\mathbb{R}^2$ by integer translates.

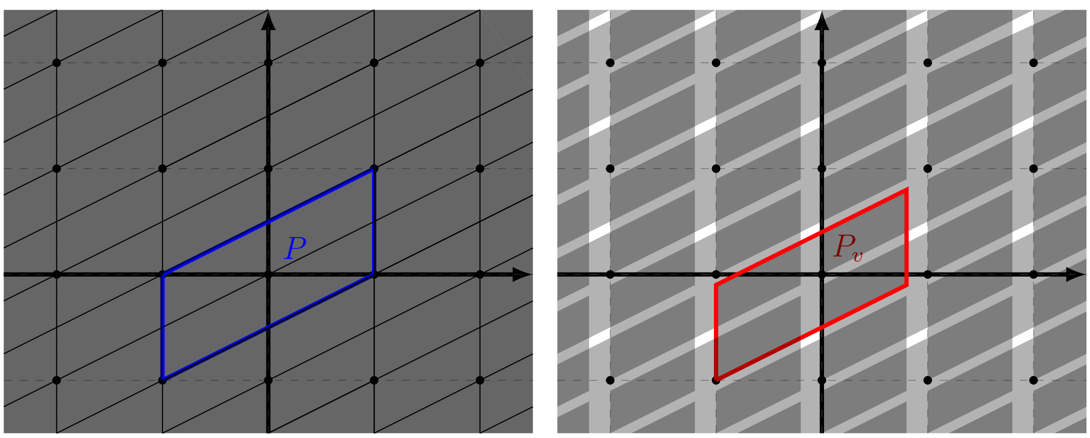

Figure 1: A parallelogram P and all of its integer translates (left); P_v failing to cover $\mathbb{R}^2$ by integer translates for $v = (-\frac{1}{5}, -\frac{1}{5})$ (right)

In Figure 1, we can begin to see that there is some subtlety to this problem. Even though $P_{(-\frac{1}{5},-\frac{1}{5})}$ is just a slight perturbation of P (which does actually cover $\mathbb{R}^2$ by integer translates), we can see that $P_{(-\frac{1}{5},-\frac{1}{5})}$ does not cover $\mathbb{R}^2$ by integer translates. In fact, as we will justify later, the set of v such that P_v covers $\mathbb{R}^2$ by integer translates is the set shown in Figure 2.

While it is completely fair to ask why we would care about this seemingly random geometric problem, the surprising answer is that this question has a deep connection to a problem in algebraic geometry. In algebraic geometry, algebraists will often look at various algebraic objects associated to the set of solutions for a system of

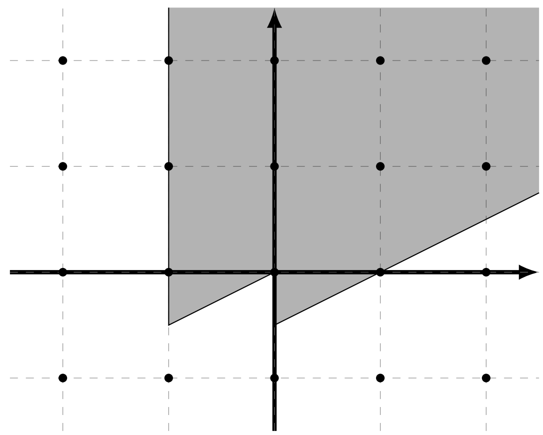

Figure 2: The set of $v \in \mathbb{R}^2$ such that P_v covers $\mathbb{R}^2$ by integer translates.

polynomial equations. This set of solutions is called a variety.

One such algebraic object is the diagonal Cartier algebra. It turns out that computing the diagonal Cartier algebra of a certain kind of variety, called a toric variety, in two dimensions is the same thing as solving the geometric problem described above.

In Chapter 1, we will introduce the necessary algebraic and geometric tools to define toric varieties. Chapter 2 will then introduce Cartier algebras and more specifically how it is they relate our initial question to toric varieties. Having then provided the technical side to the motivation for our question, Chapter 3 will focus on the solution to our initial geometric question.

Chapter 1

Putting the Geometry in Algebraic Geometry

Our main goal in this chapter is to introduce the notion of toric varieties, as they will be the central setting for our exploration of diagonal Cartier algebras. Toric varieties are a nice class of varieties because their construction naturally gives a link between algebraic objects, rings and varieties, and geometric objects, cones. To that end, in the first section we will develop some basic concepts and intuition relating to cones in Euclidean space. Once we have a firm grasp of these geometric objects, we will shift our focus to defining fundamental algebraic objects, such as monoids and rings. In the final section, we will introduce the language of algebraic geometry as a means to further relate the work done in the first two sections, prove some core algebraic geometric results, and establish a definition of toric varieties.

1.1 The Geometry of Cones

Fundamental to the definition of toric rings are cones.

Definition 1.1.1. Let $A = \{v_1, \ldots, v_m\}$ be a finite set of vectors in $\mathbb{R}^n$. Then we say the set

$$\sigma(A) = \left\{ \sum_{i=1}^{m} \lambda_i v_i \,\middle|\, \lambda_i \in \mathbb{R}, \lambda_i \geq 0 \right\}$$

is the polyhedral cone generated by A and we will refer to the v_i as the generators of σ. If the set A is clear from context or is unspecified, we will instead write σ instead of $\sigma(A)$.

Let us now consider some examples of cones.

Example 1.1.2. Let $A = \{(0,1),(1,0)\}$ and $B = \{(0,1),(1,1)\}$. Using our definition of $\sigma(A)$, we can see that the set of all non-negative combinations of A, i.e. $\sigma(A)$, is $\{(a,b)|0 \leq a, b\}$. If we look at the set B, then we can see that $\sigma(B)$ can be written as the set $\{(a,b)|0 \leq a \leq b\}$. Using the explicit forms we have found for both $\sigma(A)$ and $\sigma(B)$, we can then represent both of these cones graphically as is shown in Figure 1.1.

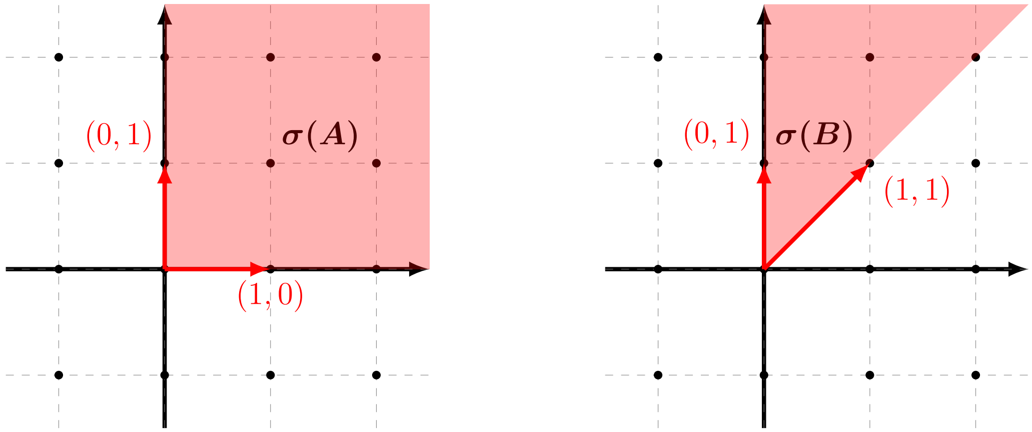

Figure 1.1: $\sigma(A)$ (left) and $\sigma(B)$ (right) pictured graphically

For our purposes, we will only consider cones that have the following properties.

Definition 1.1.3. We say that a cone σ is rational if all the generators v_i are elements of the lattice $N \cong \mathbb{Z}^n$.

Definition 1.1.4. We say that σ is strongly convex if it does not contain any straight lines going through the origin.

Since these properties will be very important as we move forward, it will be helpful to gain some intuition for cones with these conditions. As we can see in Figure 1.1, both of the cones $\sigma(A)$ and $\sigma(B)$ from Example 1.1 provide examples of rational, strongly convex cones. While these examples work really well to illustrate rational cones, the strongly convex condition is best exemplified through a non-example.

Example 1.1.5. Consider the set $C = \{(1, \frac{\sqrt{2}}{2}), (-1, -\frac{\sqrt{2}}{2}), (0, 1)\}$. While it is clear graphically that $\sigma(C)$ is not strongly convex almost immediately, as seen in figure 1.2, we can also verify that $\sigma(C)$ is not strongly convex using our definition of a cone. To do this, first note that the set $\sigma(C)$ takes the form

$$\sigma(C) = \{(x, y) | x, y \in \mathbb{R} \text{ and } y \geq 0\}.$$

From here, we can see that $\sigma(C)$ is not strongly convex because it contains the x-axis which goes through the origin. Additionally, we can also see that $\sigma(C)$ is not rational because the slope of the edge of $\sigma(C)$ is irrational, and thus the line will never contain any lattice points.

From here, the next important concept we need to introduce is that of duality. Because our toric varieties will be defined with the dual of a strongly convex rational cone, it will be helpful to introduce dual cones. However, before we can understand what dual cones are we first will need to understand what a dual vector space is.

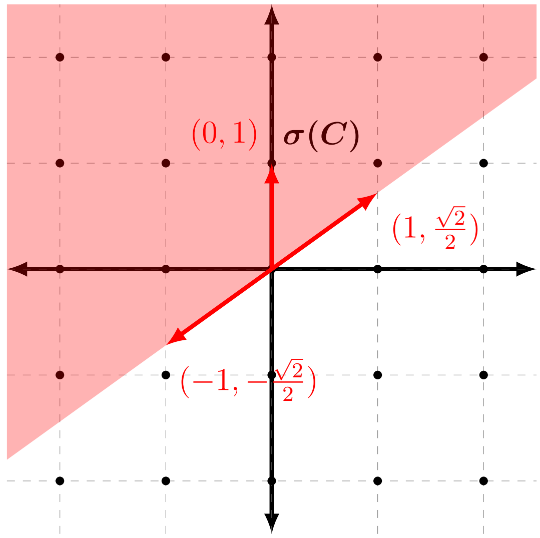

Figure 1.2: The non-rational, non-strongly convex cone $\sigma(C)$ from Example 1.1.5

Definition 1.1.6. Let V be a finite-dimensional vector space over a field k. Then the dual vector space of V, written V^*, is the set

$$V^* = \{f : V \to k \mid f \text{ is a linear map}\}.$$

Remark 1.1.7. Note that we will sometimes denote V^* as $\mathrm{Hom}_K(V, k)$ though we will save the definition of Hom for Chapter 2.

Since it is non-obvious that V^* is a vector space, consider the following proposition.

Proposition 1.1.8. *V^* is a k-vector space.*

Proof. To see that V^* is closed under addition and scalar multiplication, consider $f, g \in V^*$ and some $a \in k$. Then we need to check that $f + ag$ is a linear map. To see this, take $v, v' \in V$ and $b \in V$. Then we can see that

$$\begin{aligned}(f + ag)(v + bv') &= f(v + bv') + ag(v + bv') \\ &= f(v) + bf(v') + ag(v) + (ab)g(v') \\ &= f(v) + ag(v) + bf(v') + (ab)g(v') \\ &= (f + ag)(v) + b(f + ag)(v')\end{aligned}$$

and it follows that $f + ag \in V^*$. Additionally, because the 0 function is linear we can see that 0 acts as the additive identity for V^*. From here, all that remains to check that V^* is a vector space is that V^* satisfies the distributivity property. To see this, take $a, b \in k$ and $f, g \in V^*$. Then we can see for all $v \in V$ that

$$(a + b)f(v) = f((a + b)v) = f(av + bv) = f(av) + f(bv) = af(v) + bf(v)$$

and

$$a(f(v) + g(v)) = f(av) + g(av) = af(v) + ag(v).$$

Thus we have shown that V^* is a k-vector space. $\square$

While V^* may seem abstract, once we choose the correct basis V^* it will become clear that V^* is quite easy to work with.

Definition 1.1.9. Let $B = \{v_1, \ldots, v_n\}$ be a basis for some vector space V. Then $v_i* \in V^*$ are the linear maps defined on B ,and extended linearly to the rest of V, defined such that

$$v_i^*(v_j) = \begin{cases} 1 & i = j; \\ 0 & i \neq j. \end{cases}$$

Proposition 1.1.10. *Let V be an n-dimensional k-vector space with basis $\{v_1, \ldots, v_n\}$. Then V^* is a n-dimensional k-vector space with basis $\{v_1^*, \ldots, v_n^*\}$.*

Proof. Since we have already checked that V^* is a vector space in Proposition 1.1.8, all that remains for us to check is that $\{v_1^*, \ldots, v_n^*\}$ is a basis for V^*. For linear independence, consider some linear combination

$$\lambda_1 v_1^* + \lambda_2 v_2^* + \ldots \lambda_n v_n^* = 0.$$

Evaluating both sides at v_i, we get

$$\lambda_1 v_1^*(v_i) + \lambda_2 v_2^*(v_i) + \ldots \lambda_n v_n^*(v_i) = \lambda_i v_i^*(v_i) = \lambda_i = 0(v_i) = 0.$$

Thus we have shown that $\lambda_i = 0$ for all $0 \leq i \leq n$ and we have checked that $\{v_1^*, \ldots, v_n^*\}$ is a linearly independent set. To see that $\{v_1^*, \ldots, v_n^*\}$ is a spanning set, consider a linear function $f \in V^*$. Evaluating f and v_i, we then can see that $f(v_i) = \mu_i$. Using these values of μ_i as coefficients, we get the linear combination of basis elements $\sum_{j=1}^n \mu_j v_j^*$. To check that this linear combination is in fact equal to f, take some arbitrary element $v = \sum_{i=1}^n \lambda_i v_i$ of V. Then we can see by the linearity of f that

$$f(v) = \sum_{i=1}^n \lambda_i f(v_i) = \sum_{i=1}^n \lambda_i \mu_i = \sum_{i=1}^n \mu_i v_i^*(\lambda_i v_i) = \sum_{i=1}^n \mu_i v_i^*(v).$$

With this we have shown that $\sum_{j=1}^n \mu_j v_j^* = f$ by consequence that $\{v_1^*, \ldots, v_n^*\}$ is a spanning set. Thus we know that $\{v_1^*, \ldots, v_n^*\}$ forms a basis for V^* and that V^* must be n-dimensional □

Remark 1.1.11. In the case where V is n-dimensional, we know by the previous proposition that both V and V^* are n-dimensional and so $V \cong V^*$.

With duality understood for vector spaces, we can now define the dual of a cone.

Definition 1.1.12. Let σ be a cone in $\mathbb{R}^n$. Then we can define the dual cone of σ to be

$$\check{\sigma} = \{u \in (\mathbb{R}^n)^* | \langle u, v \rangle \geq 0 \ \ \forall v \in \sigma\}.$$

Example 1.1.13. If we let e_1, e_2 be the elements of the standard basis for our lattice N, then we can see in the following figure a picture of the cone generated by $e_1 + 2e_2$ and e_1 as well as its dual cone. For the dual cone, we will let e_1^* and e_2^* denote the basis for the lattice M.

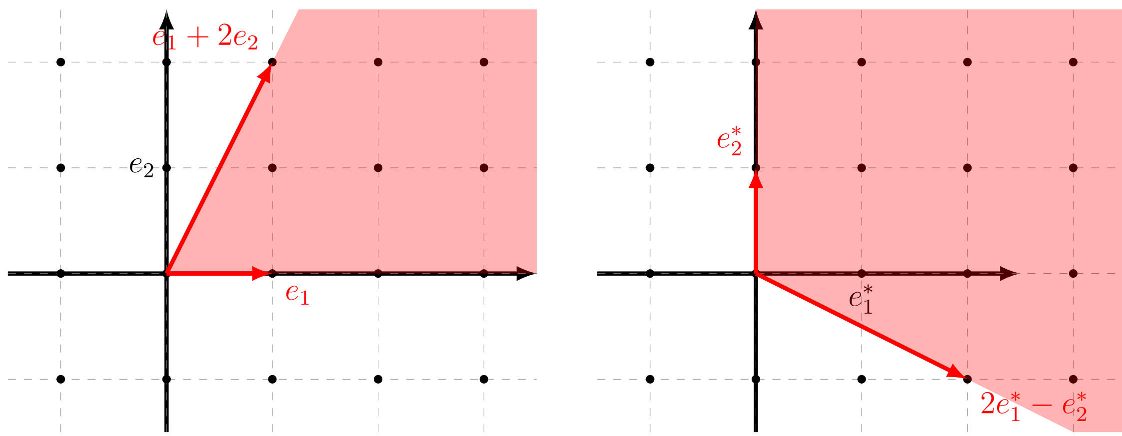

Figure 1.3: An example of a cone σ (left) and its dual cone $\check{\sigma}$ (right) in $\mathbb{R}^2$

In keeping with this duality theme and our desire to work with rational cones, it would be useful to also develop a notion of dual lattice. Let us now consider the lattice $N = \mathbb{Z}^n$. Then we can define our dual lattice to be

$$M := \left\{ \sum_{i=1}^{n} a_i e_i^* \,|\, a_i \in \mathbb{Z} \right\} \cong \mathbb{Z}^n.$$

Remark 1.1.14. To those with knowledge of Hom sets, it could be helpful to note that the dual lattice of N takes the form $M := \operatorname{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$. For those unfamiliar with Hom sets, we will define them in chapter 2.

With this notion of a dual lattice defined, we can now observe the following property having to do with rational cones.

Proposition 1.1.15. *Suppose that σ is a rational cone with respect to the lattice N. Then $\check{\sigma}$ is also a rational cone with respect to the lattice M.*

Proof. Let $\sigma(\{v_1, \ldots, v_m\})$ be a rational cone and so we get that each $v_i \in \mathbb{Z}^n$. Then for each v_i we can see that the sets

$$F_i := \{v^* \in V^* \,|\, v^*(v_i) = 1\}$$

are hyperplanes that bound $\check{\sigma}$. If we want the generating vectors of our dual cone, we know that these generating vectors will be the vectors found at the intersections of these half-planes. Further, because we can arbitrarily scale our generating vectors by any $\lambda \in \mathbb{R}$ and we will get the same cone, to check that $\check{\sigma}$ is a rational cone it will suffice to show that the intersection of $(n-1)$-many F_i sets is either empty or contains a lattice point of M. To see this, consider some non-empty intersection of $n-1$ of these F_i sets. Without loss of generality, since the order of our $\{v_1, \ldots, v_m\}$ doesn't matter we can assume we are considering the set

$$\bigcap_{i=1}^{n-1} F_i = \{v^* \in V^* \,|\, v^*(v_j) = 1 \text{ when } 1 \leq j \leq n-1\}.$$

However, considering $v^* \in V^*$ as linear combinations of the basis $\{e_1^*, \ldots, e_n^*\}$, we can rewrite this set as

$$\bigcap_{i=1}^{n-1} F_i = \left\{ \sum_{i=1}^{n} \lambda_i e_1^* \in V^* \mid \langle (\lambda_1, \ldots, \lambda_n), v_j \rangle = 1 \text{ when } 1 \leq j \leq n-1 \right\}$$

where $\langle \_, \_ \rangle$ denotes the standard inner product in $\mathbb{R}^n$. Since our $v_j \in \mathbb{Z}^n$ because σ is a rational cone, it then follows when we solve the system of equations our previous interpretation uses to define $(\lambda_1, \ldots, \lambda_n)$ that we will find some $(\lambda_1, \ldots, \lambda_n) \in \mathbb{Q}^n$ such that $v' = \sum_{i=1}^{n} \lambda_i e_1^* \in \bigcap_{j=1}^{n-1} F_j$ as desired. Scaling v' by the least common multiple of the denominators of all λ_i, we then find that

$$\bigcap_{i=1}^{n-1} F_i \cap M \neq \emptyset$$

and thus that $\check{\sigma}$ is rational with respect to the lattice M. $\square$

With this property of dual cones, as we introduce new algebraic concepts relating rational cones to certain algebraic objects we will no longer have to worry about our definitions breaking down when we wish to apply them to $\check{\sigma}$ rather than σ. Without this worry, we can now turn our attention towards the algebraic concepts underpinning toric varieties.

1.2 Monoids and Cone Algebra

In this section, we will provide the basic algebra that will serve as the foundation for the algebraic geometry we will discuss later. More specifically, in this section we will provide a definition for monoids and highlight some important links between monoids and the geometric objects we presented in the previous section. At the end of this section we will then be completely prepared to start learning the algebraic geometry necessary to define toric varieties.

To start working our way through this algebraic landscape, let us begin by defining a monoid.

Definition 1.2.1. Let S be a set equipped with an associative operation $+ : S \times S \to S$. Then we say that S is a monoid if $+$ is commutative, has an identity, and satisfies the cancellation property,

$$s + t = s' + t \implies s = s' \text{ for any } s, s' \text{ and } t \in S.$$

These objects are of use to us because whenever we have a cone σ and some lattice N, $\sigma \cap N$ is a monoid.

Much like in groups there is a notion of generators of monoids. Given this, we say that a monoid S is finitely generated if there exist elements $a_1, \ldots, a_n \in S$ such that any $s \in S$ can be written in the form

$$s = \lambda_1 a_1 + \ldots \lambda_n a_n$$

with $\lambda_i \in \mathbb{Z} \geq 0$. Additionally, we say that our $a_1, \ldots, a_n$ are generators of the monoid.

With this, we can then begin to bridge the gap between geometry and algebra by showing that $\sigma \cap N$ is a finitely generated monoid for any rational cone σ.

Lemma 1.2.2. *If σ is a strongly convex rational polyhedral cone, then $\sigma \cap N$ is a finitely generated monoid.*

Proof. Let $v_1, \ldots, v_n$ be the vectors defining our cone σ. First we will verify that $\sigma \cap N$ is a monoid. To see this, we can immediately note that the 0 vector is contained in $\sigma \cap N$ and acts as the identity element since our operation is addition. Similarly, since we are working with addition, which is known to be associative, all that remains for us to check is that $\sigma \cap N$ is closed under addition. To check this, consider $\sum \lambda_i v_i, \sum \mu_i v_i \in \sigma \cap N$. Then since these sums are both in $\sigma \cap N$, our definition of a cone tells us that $\lambda_i \geq 0$ and $\mu_i \geq 0$ for all $1 \leq i \leq n$. This then implies that $\mu_i + \lambda_i \geq 0$ and thus that $\sum(\lambda_i + \mu_i)v_i \in \sigma$. Since lattices are closed under addition, it then follows that $\sum(\lambda_i + \mu_i)v_i \in \sigma \cap N$ and thus that $\sigma \cap N$ is a monoid.

To see that $\sigma \cap N$ is finitely generated, we first note that the set $\{\sum r_i v_i,\ 0 \leq r_i \leq 1\}$ is bounded, and thus it follows that there are only finitely many points in the set $\{\sum r_i v_i,\ 0 \leq r_i \leq 1\} \cap N$. From here, for any $v \in \sigma \cap N$ by our definition of σ we can write $v = \sum(n_i + s_i)v_i$ where $n_i \in \mathbb{Z}_{\geq 0}$ and $0 \leq s_i \leq 1$. Then since $\sum n_i v_i \in N$ it follows that $\sum s_i v_i \in N$. This then tells us that $\sigma \cap N$ will be finitely generated by $\{v_1, \ldots, v_n\} \cup \{\sum r_i v_i,\ 0 \leq r_i \leq 1\}$ and we have our result. $\square$

Example 1.2.3. To see an example of such a monoid, consider the cone $\sigma(\{e_1, e_1 + 2e_2\})$ presented in Example 1.1.13. Then looking at Figure 1.4 we can observe that the monoid $\sigma \cap N$ is generated by the vectors $\{e_1, e_1 + e_2, e_1 + 2e_2\}$.

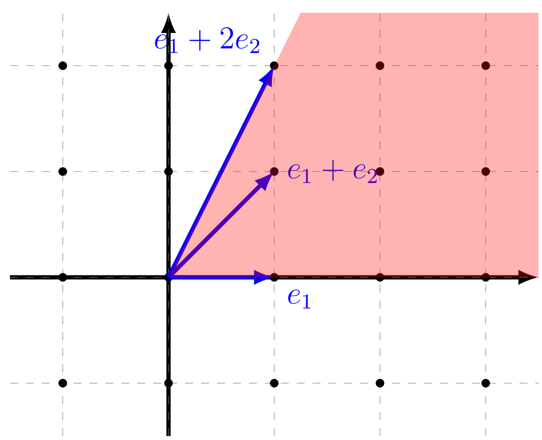

Figure 1.4: The cone σ along with generators of $\sigma \cap N$ highlighted in blue (Example 1.3.3)

With these results, we can then define a special ring R_σ, that will serve as the last algebraic tool necessary for defining affine toric varieties. To define this ring R_σ, we will establish a method of generating a ring from a rational cone σ. To understand how we construct R_σ, we will first introduce a helpful motivating result of a similar flavor and then provide our construction of R_σ.

Let $\mathbb{C}[z, z^{-1}] = \mathbb{C}[z_1, \ldots, z_n, z_1^{-1}, \ldots, z_n^{-1}]$ denote the Laurent polynomial ring. Then we can establish a connection between the lattice $\mathbb{Z}^n$ and $\mathbb{C}[z, z^{-1}]$.

Proposition 1.2.4. *Let $\phi : \mathbb{Z}^n \to \mathbb{C}[z, z^{-1}]$ be the map defined such that for any $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$, $\phi(a) = z_1^{a_1} \cdots z_n^{a_n} =: z^a$. Then ϕ is an isomorphism between the additive group $\mathbb{Z}^n$ and the multiplicative group of monic Laurent polynomials.*

Proof. To show that ϕ is an isomorphism we will note that ϕ is clearly a bijection. This means we only need to check that ϕ is a group homomorphism and we will be done. To do so, we first note that $\phi(0) = z^0$ and thus that ϕ preserves the identity. Now take some $a, b \in \mathbb{Z}^n$. Then

$$\phi(a)\phi(b) = (z_1^{a_1} z_2^{a_2} \ldots z_n^{a_n})(z_1^{b_1} z_2^{b_2} \ldots z_n^{b_n}) = (z_1^{a_1+b_1} z_2^{a_2+b_2} \ldots z_n^{a_n+b_n}) = \phi(a+b)$$

and we have seen that ϕ preserves our group operation and is thus a group homomorphism.

$\square$

With this result as well as the natural relation between lattices and $\mathbb{Z}^n$, we can now put Lemma 1.2.2 to use for constructing R_σ, the objects we need to be able to suitably define our affine toric varieties.

Definition 1.2.5. The support of a Laurent polynomial $f = \sum_{\text{finite}} \lambda_a z^a$ is defined to be

$$\text{supp}(f) = \{a \in \mathbb{Z}^n : \lambda_a \neq 0\}.$$

With this notion of support, for a strongly convex rational cone σ we can define the following important set

$$R_\sigma = \{f \in \mathbb{C}[z, z^{-1}] \mid \text{supp}(f) \subset \check{\sigma} \cap M\}.$$

For convenience, later in the thesis we will choose to denote R_σ as $k[\check{\sigma} \cap M]$. With the notation out of the way, we can see that R_σ satisfies the following important property.

Proposition 1.2.6. *For a strongly convex lattice cone σ, R_σ is a ring.*

Proof. Since $R_\sigma \subset \mathbb{C}[z, z^{-1}]$, to verify that R_σ is a ring it will suffice to verify that R_σ is a subring of $\mathbb{C}[z, z^{-1}]$. This is simpler because it allows us to only check that R_σ is closed under addition, closed under multiplication, and contains the multiplicative identity rather than checking that all of the ring axioms are satisfied. Starting with proving that R_σ is closed under addition, let us consider any $f, g \in R_\sigma$. We can then observe from our definition of the support of Laurent polynomial that $\text{supp}(f+g) \subseteq \text{supp}(f) \cup \text{supp}(g)$. Since $\text{supp}(f), \text{supp}(g) \subset \check{\sigma} \cap M$ by our definition of R_σ, it then follows from our earlier observation that $\text{supp}(f+g) \subset \check{\sigma} \cap M$ and thus that $f+g \in R_\sigma$ as desired. To verify that R_σ is closed under multiplication we now need to verify that $f + g \in R_\sigma$. To see this, note that f takes the form

$$f = \sum_{a \in \text{supp}(f)} \lambda_a z^a$$

and similarly, g takes the form

$$g = \sum_{b \in \text{supp}(g)} \mu_b z^b.$$

Using these forms for f and g, we can clearly more easily compute fg and we see that

$$fg = \sum_{a \in \text{supp}(f)} \sum_{b \in \text{supp}(g)} \lambda_a \mu_b z^{a+b}.$$

Since by assumption we know $a, b \in \check{\sigma} \cap M$ for all $a \in \text{supp}(f)$ and $b \in \text{supp}(g)$, by Lemma 1.2.2 it follows that $a+b \in \check{\sigma} \cap M$. Thus we have shown that $\text{supp}(fg) \subset \check{\sigma} \cap M$ and we have verified that R_σ is closed under multiplication.

Lastly, to check that R_σ contains the multiplicative identity of $\mathbb{C}[z, z^{-1}]$ we will again make use of Lemma 1.2.2. By the aforementiod lemma, we know that $\check{\sigma} \cap M$ is a monoid and thus that $0 \in \check{\sigma} \cap M$. This then tells us that $\text{supp}(1) = \text{supp}(z_1^0 z_2^0 \dots z_n^0) = \{0\} \subset \check{\sigma} \cap M$ and thus the multiplicative identity of $\mathbb{C}[z, z^{-1}]$ is in R_σ. With this, we have verified that R_σ is a subring of $\mathbb{C}[z, z^{-1}]$ and thus that R_σ is a ring. □

Example 1.2.7. Building off of our previous examples, we will find R_σ for the σ presented in Example 1.1.13. To find R_σ, we will first recall by Lemma 1.2.2 that $\check{\sigma} \cap M$ is finitely generated. In fact, it is not so hard to see that the set $\{e_1^*, e_2^*, 2e_1^* - e_2^*\}$ is a generating set for $\check{\sigma} \cap M$. This observation allows us to then compute R_σ and to see that $R_\sigma = \mathbb{C}[x, y, x^2 y^{-1}]$.

1.3 Algebraic Geometry and Toric Varieties

In order to achieve our goal of defining Toric Varieties, and gaining a way to leverage geometric concepts to generate algebraic results, we first will need to understand some core algebraic geometry concepts.
For the rest of this section, let $\mathbb{C}[z] = \mathbb{C}[z_1, \dots, z_k]$ be the ring of polynomials in k variables over the field $\mathbb{C}$.

Definition 1.3.1. If $E = \{f_1, \dots, f_r\} \subset \mathbb{C}[z]$, then

$$V(E) = \{x \in \mathbb{C}^k \,|\, f_1(x) = \cdots = f_r(x) = 0\}$$

is called the affine algebraic set defined by E. If I is the ideal generated by E, then we say that $V(I) = V(E)$.

Less formally, the affine algebraic set defined by some $E = \{f_1, \dots, f_n\} \subset \mathbb{C}[z]$ is the set of solutions to the system of equations $f_1(x) = \cdots = f_r(x) = 0$.

Definition 1.3.2. Let $X \subset \mathbb{C}^k$. Then

$$I(X) = \{f \in \mathbb{C}(z) \,|\, \forall x \in X,\ f(x) = 0\}$$

is called the vanishing ideal of X

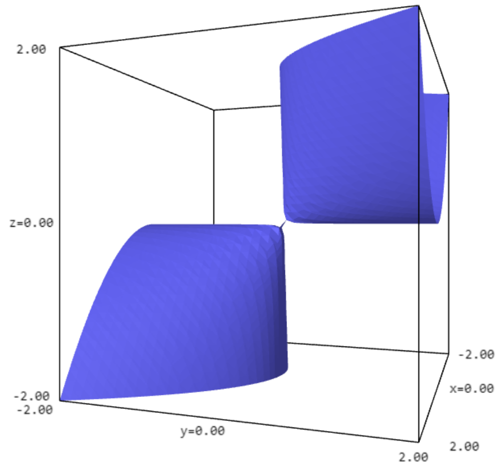

Figure 1.5: X_σ graphed for Example 1.3.7

Proposition 1.3.3. *For any $X \subset \mathbb{C}^k$, $I(X)$ is an ideal of $\mathbb{C}[z]$.*

Proof. Let $X \subset \mathbb{C}^k$ and $I(X)$ be the vanishing ideal of X. Then we can see that for any $f, g \in I(X)$ that $f + g \in I(X)$ since $f + g(x) = f(x) + g(x) = 0$ for all $x \in X$. Additionally, we can see that the zero function is clearly in $I(X)$ and we know that $I(X)$ is an Abelian subgroup of $\mathbb{C}[z]$. To see that $I(X)$ is an ideal, it then suffices to show that for $f \in I(X)$ and $g \in \mathbb{C}[z]$ that the function $h(x) := f(x)g(x) \in I(X)$. To see this, we simply observe for all $x \in X$ that $h(x) = f(x)g(x) = 0g(x) = 0$ and thus $h(x) \in I(X)$ as desired. $\square$

With this vanishing ideal definition, we can then notice that for any $x \in \mathbb{C}^k$, $I(\{x\})$ is a maximal ideal we denote $\mathcal{M}_x$. With this language of maximal ideals, we can then introduce a key result from algebraic geometry that will allow us to bring together the geometric and algebraic objects we established in the previous sections.

Theorem 1.3.4 ([Bra01], Theorem 2.1)**.** *Every maximal ideal in $\mathbb{C}[z]$ can be written $\mathcal{M}_x$ for a point x.*

As for why we find this useful, this result shows that there is a one-to-one correspondence between points in $\mathbb{C}^k$ and maximal ideals of $\mathbb{C}[z]$. While this result does not yet provide a direct application for our algebraic and geometric objects, the following corollary of it does.

Corollary 1.3.5 ([Bra01],Corollary 2.2)**.** *Let V be an affine algebraic set and let $R_V = \mathbb{C}[z]/I(V)$. Then there is a one-to-one correspondence*

$$V \longleftrightarrow \{\mathcal{M} \subset R_V \,|\, \mathcal{M} \text{ maximal ideal}\} =: \mathrm{Spec(R_V)}$$

Applying this result to our rings R_σ, where σ is a rational, strongly convex cone, then gives us a natural method to give our cones algebraic representations that we can work with. Seeking to study these specific objects which arise from our cones, we get the following definition of affine toric varieties:

Definition 1.3.6. Let σ be a rational, polyhedral, strictly convex cone. Then the affine toric variety corresponding to σ is $X_\sigma := \mathrm{Spec}(\mathrm{R}_\sigma)$.

To see how we can leverage these results, we will look at some examples of different toric varieties and how we can construct our aforementioned one-to-one correspondence.

Example 1.3.7. In keeping with our running example, lets again look at the cone $\sigma = \sigma(\{e_1, e_1+2e_2\})$. Then in Example 1.2.7 we saw that $R_\sigma = \mathbb{C}[x, y, x^2y^{-1}]$. Setting $z = x^2y^{-1}$, we can then see that R_σ can also be represented as

$$R_\sigma = \mathbb{C}[x, y, z] = \mathbb{C}[z_1, z_2, z_3]/(z_2z_3 - z_1^2).$$

Taking this ideal $I_\sigma = (z_2z_3 - z_1^2)$ from our new representation of R_σ, we can then see that the toric variety X_σ takes the form

$$X_\sigma = V(I_\sigma) = \{x = (x_1, x_2, x_3) \in \mathbb{C}^3 \mid x_2x_3 = x_1^2\}.$$

Chapter 2

Putting the Algebra in Commutative Algebra

Having defined toric varieties and showcased how they can serve as a bridge between algebra and geometry, we will now shift our perspective to the world of commutative algebra. In this chapter, we will start by providing a basic overview of the requisite language and definitions required to work in our new setting. Once we are able to, we will then introduce the diagonal Cartier algebras that we wish to study and see how, in the context of toric varieties, these Cartier algebras possess a more workable interpretation.

2.1 Modules and Commutative Algebra

In this section, our primary goal will be to introduce the key concepts from commutative algebra that will be necessary for setting up our future question. To start, we will begin by defining a tool that will not only set the stage for the section to come but will also play a key role throughout the entirety of the chapter. However, before we can define what will be our main tool, the Frobenius map, we first need to establish what types of rings we will be considering.

Definition 2.1.1. Let R be a ring. Then we say the characteristic of R is the smallest number of times one must add 1 (the multiplicative identity) to itself in order to get 0 (the additive identity). If such a sum does not exist, we say R has characteristic zero.

For the rest of the section, the rings we consider will be rings of positive characteristic. Another important property in commutative algebra is the Noetherian property of While this may seem like an odd choice, it will be necessary for ensuring that our main tool the Frobenius map behaves in the way that we need it to.

Definition 2.1.2. Let R be a commutative Noetherian ring of characteristic $p > 0$. Then we can define the Frobenius map as follows:

$$\begin{aligned} F :& R \to R \\ & x \mapsto x^p \end{aligned}$$

Similarly, we let F^e denote the e^{th} iterate of the Frobenius map. That is, $F^e(x) = F(F \cdots (F(x))) = x^{p^e}$.

While it may not seem interesting upon first inspection, because we are working with commutative rings of positive characteristic the Frobenius map has a lot of interesting properties. One such property that we can verify now is that the Frobenius map is a ring homomorphism.

Proposition 2.1.3. *Let R be a Noetherian ring of characteristic $p > 0$ and F be the Frobenius map. Then F is a ring homomorphism.*

Proof. To verify that F is a ring homomorphism, we need to check that F is unit preserving and respects both multiplication and addition. Starting with the unit preserving property, we can see that $F(1_R) = 1_R^p = 1_R$ as desired. To check that multiplication is preserved first take some $a, b \in R$. Then we can see $F(ab) = (ab)^p$ and, because R is commutative, it follows that $(ab)^p = a^p b^p = F(a)F(b)$. Thus we see that $F(ab) = F(a)F(b)$ as desired. Finally, to see that F respects addition, recall that in a ring of characteristic $p > 0$ $(a + b)^p = a^p + b^p$ for all $a, b \in R$. With this we can then see $F(a + b) = (a + b)^p = a^p + b^p = F(a) + F(b)$. $\square$

However, before we can see some other useful aspects of the Frobenius map we will need to define a module.

Definition 2.1.4. Let R be a ring with multiplicative identity 1. Then a module M is an abelian group, whose operation we will denote with $+$, together with an operation $\cdot : R \times M \to M$, which we will refer to as scalar multiplication, that satisfies the following conditions for all $r, s \in R$ and $x, y \in M$:

1. $r \cdot (x + y) = r \cdot x + r \cdot y$,
2. $(r + s) \cdot x = r \cdot x + s \cdot x$,
3. $(rs) \cdot x = r \cdot (s \cdot x)$,
4. $1 \cdot x = x$.

We will call such a module M a left R-module and note that a similar set of conditions can define a right module whose operation takes inputs from $M \times R$ rather than $R \times M$. Additionally, we say that $N \subseteq M$ is a submodule of M if it is closed under addition and scalar multiplication

Remark 2.1.5. While this definition may seem new, in the case where R is a field our modules are actually just vector spaces.

To better understand the definition of a module, we can look at some key examples of modules that we can construct using the Frobenius map that will be important to us later.

Proposition 2.1.6. *Consider the set $F^e_* R := \{F^e_* r \,|\, r \in R\}$. Then $F^e_* R$ is an R-module with R-module structure given by $sF^e_* r = F^e_* s^{p^e} r$ for all $s, r \in R$.*

Proof. To see that $F_*^e R$ is an R-module, take some $x, y, s, r \in R$. Then we can see that

$$\begin{aligned} sF_*^e(x+y) &= F_*^e\left(s^{p^e}(x+y)\right) \\ &= F_*^e(s^{p^e}x + s^{p^e}y) \\ &= F_*^e(s^{p^e}x) + F_*^e(s^{p^e}y) \\ &= sF_*^e x + sF_*^e y \end{aligned}$$

and thus our scalar multiplication distributes as we need. Next, to check that scalar multiplication respects addition in R we do the following computation:

$$\begin{aligned} (s+r)F_*^e x &= F_*^e(s+r)^{p^e}x \\ &= F_*^e(s^{p^e} + r^{p^e})x \\ &= F_*^e(s^{p^e}x + r^{p^e}x) \\ &= F_*^e s^{p^e}x + F_*^e r^{p^e}x \\ &= sF_*^e x + rF_*^e x. \end{aligned}$$

With this, we have shown that our scalar multiplication respects addition in the underlying ring R so now we need to check the same for multiplication in the underlying ring R. To do so, we can see that

$$\begin{aligned} (sr)F_*^e x &= F_*^e(sr)^{p^e}x \\ &= F_*^e s^{p^e} r^{p^e} x \\ &= s\left(F_*^e r^{p^e} x\right) \\ &= s\left(rF_*^e x\right). \end{aligned}$$

From here all that remains is to check is that scalar multiplication by the identity acts as an identity element. This follows directly though because $1F_*^e x = F_*^e 1^{p^e} x = F_*^e x$. □

Example 2.1.7. Let $R = \mathbb{F}_2[x, y]$. Then we will show that F_*R is really just the free R-module generated by $B = \{F_*1, F_*x, F_*y, F_*xy\}$. To see these basis elements span, it suffices to show that for all $F_*x^n y^m \in F_*R$ there exists some $r \in R$ and basis element $b \in B$ such that $rb = F_*x^n y^m$. To see this, consider some arbitrary $x^n y^m \in R$. Then we can compute that

$$\begin{aligned} x^n y^m F_*1 &= F_*x^{2n}y^{2m} \\ x^n y^m F_*x &= F_*x^{2n+1}y^{2m} \\ x^n y^m F_*y &= F_*x^{2n}y^{2m+1} \\ x^n y^m F_*xy &= F_*x^{2n+1}y^{2m+1} \end{aligned}$$

and it is clear to see that any $F_*x^n y^m \in F_*R$ can be written as just rb with $r \in R$ and $b \in B$. Similarly, this same computation shows that our basis elements cannot be obtained from each other since the parities of the exponents will always be different depending on which basis element we choose. Thus now know that F_*R is the free R-module generated by the points F_*1, F_*x, F_*y, and F_*xy.

Example 2.1.8. Let k be a field of positive characteristic $p > 0$. Then for a toric ring $R = k[\check{\sigma} \cap \mathbb{Z}^d]$ we can see that $F^e_* R = k[\check{\sigma} \cap \frac{1}{p^e}\mathbb{Z}^d]$. This is worked out in the preliminaries of [Pay09].

Before we can get to our other interesting R-modules that we can construct using the Frobenius map, we need another two definitions.

Definition 2.1.9. Let V and W be R-modules. Then we say $f : V \to W$ is a module homomorphism so long as f preserves the module structures. That is, for all $x, y \in V$ and $r \in R$

$$f(x + y) = f(x) + f(y)$$

and

$$f(rx) = rf(x).$$

Definition 2.1.10. Let R and S be rings. Then we can define the group

$$\text{Hom}(R, S) := \{f : R \to S \mid f \text{ is a ring homomorphism}\}$$

with addition given by the standard addition of functions. If R and S are modules of the same ring, then we say

$$\text{Hom}(R, S) := \{f : R \to S \mid f \text{ is a module homomorphism}\}.$$

With these Hom sets defined and our previous example $F^e_* R$, we can then give two more examples of R-modules that will each play an important role in the definition of a Cartier algebra.

Definition 2.1.11. Let R be a Noetherian ring of characteristic $p > 0$. Then $\text{Hom}(F^e_* R, R)$ forms a left R-module with scalar multiplication given by

$$r \cdot (\phi(F^e_* x)) = r\phi(F^e_* x)$$

for all $r, x \in R$ and $\phi \in \text{Hom}(F^e_* R, R)$. We will denote this module $\text{Hom}_R(F^e_* R, R)$.

Definition 2.1.12. Let R be a Noetherian ring of characteristic $p > 0$. Then $\text{Hom}(F^e_* R, R)$ forms a right R-module with scalar multiplication given by

$$(\phi \cdot r)(F^e_* x) = \phi(F^e_* rx)$$

for all $r, x \in R$ and $\phi \in \text{Hom}(F^e_* R, R)$. We will refer to this as the pre-multiplication module for $\text{Hom}_R(F^e_* R, R)$.

Now that we have this R-module structures on $\text{Hom}(F^e_* R, R)$, there is one more necessary definition we need before we can talk about Cartier algebras.

Definition 2.1.13. A ring R is F-finite if $F^e_* R$ is a finitely generated R-module for some $e > 0$.

2.2 Introducing Cartier Algebras

Cartier algebras are an interesting algebraic invariant that help algebraic geometers study rings. For example, they are an essential ingredient in the proof of a subadditivity formula for test ideals of regular rings [HY03]. The diagonal Cartier algebras were then developed by Smolkin [Smo20] as a generalization of Cartier algebras meant to study potential subadditivity formula for test ideals of non-regular rings.

With some motivation behind our interest in diagonal Cartier algebras, we can now establish the algebraic setting in which we will be looking at Cartier algebras. To that end, we will assume for the rest of this section that our rings R are F-finite, Noetherian, commutative, and of positive characteristic.

With the setting established, we will now work towards defining the diagonal Cartier algebras that we care about. To start down this path, it is only natural that we start with the definition of a Cartier algebra.

Definition 2.2.1. A Cartier algebra on R is an additive Abelian group $\mathscr{C} = \bigoplus_e \mathscr{C}_e$ where $\mathscr{C}_e \subseteq \operatorname{Hom}_R(F_*^e R, R)$ is a submodule of $\operatorname{Hom}_R(F_*^e R, R)$ with respect to the module structures given in Definition 2.1.11 and Definition 2.1.12. Additionally, we also require that for all $\phi \in \mathscr{C}_e$ and $\psi \in \mathscr{C}_d$ that the maps "compose" in that

$$\phi \circ F_*^e \psi \in \mathscr{C}_{e+d}$$

where $F_*^e \psi$ is the map from $F_*^{e+d} R \to F_*^e R$ such that

$$F_*^e \psi(F_*^{e+d} x) = F_*^e \psi(F_*^d x)$$

With this definition of Cartier algebras, in order for us to define diagonal Cartier algebras we will need to introduce the concept of compatibility. Not only will this definition move us one step closer to defining our diagonal algebras, but it will also allow us to provide useful examples of Cartier algebras.

Definition 2.2.2. Let R be a ring and J an ideal of R, and let $\varphi : F_*^e R \to R$ (be a morphism?). Then we say J is compatible with φ, is φ-compatible, or that φ is compatible with J if $\varphi(F_*^e J) \subseteq J$.

Because we will be depending on this compatibility criterion quite heavily, it will useful for us to see some examples of compatibility separate from Cartier algebras before linking the two concepts.

Example 2.2.3. Let $R = \mathbb{F}_2[x, y]$ as in Example 2.1.7. Then we can consider the maps $\alpha : F_* R \to R$ defined on the basis elements of $F_* R$ as

$$\begin{aligned}
\alpha(F_* 1) &= 0, \\
\alpha(F_* x) &= 1, \\
\alpha(F_* y) &= 0, \\
\alpha(F_* xy) &= 0.
\end{aligned}$$

Then consider the ideal $\langle y\rangle$. To check that $\langle y\rangle$ is α compatible, we need to verify that $\alpha(F_*\langle y\rangle) \subseteq \langle y\rangle$. To see this, it will suffice to check that $\alpha(F_* x^i y^j)$ is divisible by y for $1 \leq j$ and $0 \leq i$. By our computations in Example 2.1.7 we already know that $\alpha(F_* x^n y^m) \neq 0$ only when n is odd and m is even. Thus we can see that

$$\alpha(F_* x^i y^j) = \begin{cases} 0 & \text{if } i \text{ is even or } j \text{ is odd} \\ x^{(i-1)/2} y^{j/2} & \text{otherwise} \end{cases}$$

and because $1 \leq j$ we know that $\alpha(F_* x^i y^j) \in \langle Y\rangle$ when $x^i y^j \in \langle y\rangle$ as desired. Thus we have shown that the ideal $\langle y\rangle$ is α-compatible.

Having seen compatibility in action, we can see a clear extension of this compatibility concept to Cartier algebras.

Definition 2.2.4. Let $\mathscr{C}$ be a Cartier algebra on R. Then we say that J is $\mathscr{C}$-compatible or that $\mathscr{C}$ is compatible with J if J is compatible with each map in $\mathscr{C}$.

Using this notion of compatibility, we can see some interesting examples of specific types of Cartier algebras.

Definition 2.2.5. Let $\mathscr{C}$ be a Cartier algebra on R compatible with some ideal $I \subseteq R$. We define the *restriction of* $\mathscr{C}$ *to* R/I, denoted $\mathscr{C}|_{R/I}$, to be the set of maps $\bigoplus_{e\geq 0} \mathscr{C}_e|_{R/I}$, where

$$\mathscr{C}_e|_{R/I} := \{\bar{\varphi} : F^e_*(R/I) \to R/I \mid \varphi \in \mathscr{C}_e\}.$$

Proposition 2.2.6 ([Smo20], Proposition 3.2)**.** *Let* $\mathscr{C}$ *be a Cartier algebra on* R *compatible with some ideal* $I \subseteq R$*. Then* $\mathscr{C}|_{R/I}$ *is a Cartier algebra on* R/I*.*

Definition 2.2.7. Let $\mathscr{C}$ be a Cartier algebra on R and let $I \subseteq R$ be an ideal. Then we define the *subalgebra compatible with* I, denoted $\mathscr{C}^{I\circlearrowleft}$, to be the set of maps $\bigoplus_{e\geq 0} \mathscr{C}_e^{I\circlearrowleft}$ where

$$\mathscr{C}_e^{I\circlearrowleft} := \{\varphi | \varphi \in \mathscr{C}_e,\, \varphi(F^e_* I) \subseteq I\}.$$

Proposition 2.2.8. *Let* $\mathscr{C}$ *be a Cartier algebra on* R *and* $I \subseteq R$ *an ideal. Then* $\mathscr{C}^{I\circlearrowleft}$ *is a Cartier algebra.*

Proof. Take some $\phi \in \mathscr{C}_e^{I\circlearrowleft}$ and $\psi \in \mathscr{C}_d^{I\circlearrowleft}$. Then to check our submodule condition we can see that for all $x \in R$ that $x\phi(F^e_* I) \subseteq xI$ and, because I is an ideal, it follows that $x\phi(F^e_* I) \subseteq I$. Similarly, again making use of the fact that I is an ideal, we can see $xI \subseteq I$ and thus $\phi(F^e_* xI) \subseteq \phi(F^e_* xI) \subseteq I$. Lastly, because I is an ideal and is closed under addition we know that for all $\phi_1, \phi_2 \in \mathscr{C}_e^{I\circlearrowleft}$ that $\phi_1(F^e_* I) + \phi_2(F^e_* I) \subseteq I$, thus showing that $\mathscr{C}_e^{I\circlearrowleft}$ satisfies our submodule condition. To verify that $\mathscr{C}^{I\circlearrowleft}$ satisfies our composition condition, we simply compute that

$$\phi \circ F^e_* \psi(F^{e+d}_* I) = \phi(F^e_* \psi(F^d_* I)) \subseteq \phi(F^e_* I) \subseteq I$$

and we see that $\phi \circ F^e_* \psi \in \mathscr{C}_{e+d}^{I\circlearrowleft}$ as desired. □

With these classes of Cartier algebras, all that we need before we can define diagonal Cartier algebras is the definition of a tensor product.

Definition 2.2.9. Let M and N be R-modules. Then the tensor product $M \otimes_R N$ is an R-module equipped with a bilinear map $\otimes : M \times N \to M \otimes_R N$ such that for each bilinear map $B : M \otimes N \to P$ there is a unique linear map $L : M \otimes_R N \to P$ makes the following diagram commute.

$$\begin{tikzcd} & M \otimes_R N \arrow[dd, dashed, "L"] \\ M \times N \arrow[ur, "\otimes"] \arrow[dr, "B"'] & \\ & P \end{tikzcd}$$

Given the abstract nature of this definition, for the sake of understanding, we will look at an example of a tensor product that will be quite important for us.

Example 2.2.10. Let $k[x]$ and $k[y]$ to be polynomial rings. Then $k[x] \otimes_k k[y] \cong k[x, y]$. To see this we simply take the isomorphism $f : k[x] \otimes k[y]$ defined on the generating elements as

$$f(x^n \otimes y^m) = x^n y^m$$

and expand f k-linearly. Then it is clear that $k[x] \otimes_k k[y] \cong k[x, y]$ and in the context of tensoring polynomial rings we will typically favor the $k[x, y]$ interpretation.

With the tensor, we can now define the diagonal ideal for a ring R.

Definition 2.2.11. Let R be a k-algebra essentially of finite type, where k is a perfect field of positive characteristic. Then $I_\Delta \subseteq R \otimes_k R$ denotes the kernel of the map $\mu : R \otimes_k R \to R$ given by $\mu(x \otimes_k y) = xy$.

With this new ideal, we can bring back our notion of compatibility to define our diagonal Cartier algebra.

Definition 2.2.12. We let $\mathscr{C}^{R \otimes_k R, I_\Delta \circlearrowleft} := \left(\mathscr{C}^{R \otimes_k R}\right)^{I_\Delta}$ denote the Cartier algebra on $R \otimes_k R$ compatible with I_Δ. We say that such maps are *compatible with the diagonal.*

Definition 2.2.13. We define the second diagonal Cartier algebra on R to be

$$\mathscr{D}^{(2)}(R) := \mathscr{C}^{R \otimes_k R, I_\Delta \circlearrowleft}|_{(R \otimes_k R)/I_\Delta}.$$

If the ring R is clear from the context we will simply denote the second diagonal Cartier algebra as $\mathscr{D}^{(2)}$.

While it may seem odd that we care about these $\mathscr{D}^{(2)}$ objects, for those interested $\mathscr{D}_e^{(2)}(R)$ is the set of the maps $\phi : F_*^e R \to R$ that admit a lifting property to $R \otimes_k R$. That is, they are the maps ϕ such that the following diagram commutes:

$$\begin{tikzcd} F^e_*(R \otimes_k R) \arrow[r, dashed] \arrow[d, "F^e_*\mu"'] & R \otimes_k R \arrow[d, "\mu"] \\ F^e_* R \arrow[r, "\phi"] & R \end{tikzcd}$$

Making use of this interesting property of $\mathscr{D}^{(2)}(R)$, in the next section we will introduce a key result re-characterizing $\mathscr{D}^{(2)}(R)$ in terms of a more workable definition when working in the toric setting.

2.3 Understanding $\mathscr{D}^{(2)}$ in the Toric Setting

Throughout the rest of the chapter, let $\sigma = \sigma(\{v_1, \ldots, v_m\})$ be a rational, strongly convex cone and then let $X = X_\sigma$ be the associated affine toric variety. With this established, throughout the rest of this chapter we will be presenting the work of Smolkin that re-characterizes $\mathscr{D}^{(2)}$ in the toric setting. Since many of the proofs are quite difficult and do not pertain to the goal of our thesis, we will primarily be presenting proof ideas and anyone interested in the proofs is encouraged to look at chapter 6 of [Smo20].

With that said, now that we are working with toric varieties we immediately have tools with which we can describe the maps in our Cartier algebra.

Definition 2.3.1. For $a \in \frac{1}{p^e}\mathbb{Z}^n$. let $\pi_a : F^e_*\mathbb{C}[z, z^{-1}] \to \mathbb{C}[z, z^{-1}]$ be the map such that

$$\pi_a(u) = \begin{cases} x^{a+u} & a + u \in \mathbb{Z}^n; \\ 0, & \text{otherwise} \end{cases}$$

Noting that $R = R_\sigma \subseteq \mathbb{C}[z, z^{-1}]$, we can then see that any map in $\operatorname{Hom}(F^e_*R, R)$ can be extended to a map in $\operatorname{Hom}(F^e_*\mathbb{C}[z, z^{-1}], \mathbb{C}[z, z^{-1}])$ that maps F^e_*R into R. To characterize these maps, we need the following definition from Payne.

Definition 2.3.2. The *anticanonical polytope of sigma* is

$$P_R = \{u \in \mathbb{R}^n | \langle u, v_i\rangle \geq -1 \text{ for } 1 \leq i \leq m\}$$

where v_i are the generating rays of σ.

With this, Payne then characterizes such maps in the following way

Proposition 2.3.3 ([Pay09], Lemma 4.1). *The set of maps π_a form a basis for $\operatorname{Hom}_R(F^e_*R, R)$ where $a \in \operatorname{int}(P_R) \cap \frac{1}{p^e}\mathbb{Z}$.*

With this context established, we can finally start presenting the recharacterization of $\mathscr{D}^{(2)}$ in the toric variety setting as well as the general idea of the proof behind it.

Theorem 2.3.4 ([Smo20], Theorem 6.4). *$\mathscr{D}^{(2)}_e(R)$ is generated by the maps π_a where $a \in \frac{1}{p^e}\mathbb{Z}^n \cap \operatorname{int}(P_R)$ and the interior of $P_R \cap (a - P_R)$ contains a representative of each equivalence class in $\frac{1}{p^e}\mathbb{Z}^n/\mathbb{Z}^n$.*

In order to best present the proof of this theorem, it will be important that we remember what exactly $\mathscr{D}_e^{(2)}(R)$ is a set of. First, as we argued before, because any maps in $\mathrm{Hom}(F_*^e R, R)$ can be extended to maps in $\mathrm{Hom}(F_*^e \mathbb{C}[z, z^{-1}], \mathbb{C}[z, z^{-1}])$ to gain some intuition we will begin by working with $\mathbb{C}[z, z^{-1}]$. For the sake of notational convenience, let $T = \mathbb{C}[z, z^{-1}]$. Then, as we saw at the end of the previous section, we know that $\mathscr{D}_e^{(2)}(R)$ is the set of maps $\phi : F_*^e R \to R$ that are compatible with the ideal I_Δ and also make the following diagram commute.

$$\begin{tikzcd} F_*^e(T \otimes_k T) \arrow[r, dashed] \arrow[d, "F_*^e \mu"'] & T \otimes_k T \arrow[d, "\mu"] \\ F_*^e T \arrow[r, "\phi"] & T \end{tikzcd}$$

Utilizing the conditions given by the diagram, one can prove, as can be seen in [Smo20], the following technical Lemma providing conditions for which maps are compatible with I_Δ.

Lemma 2.3.5 ([Smo20], Lemma 6.5)**.** *Let $\phi = \sum c_{a,a'} \pi_a \otimes_k \pi_{a'}$ be a map in $\mathrm{Hom}_{k[T \times T]}(F_* k[T \times T], k[T \times T])$. Then ϕ is compatible with I_Δ if and only if for all equivalence classes $[u_1], [u_2] \in \frac{1}{p^e}\mathbb{Z}^n/\mathbb{Z}^n$, we have*

$$\sum_{a \in [u_1]} c_{a,d-a} = \sum_{b \in [u_2]} c_{b,d-b}$$

for all $d \in \frac{1}{p^e}\mathbb{Z}^n$.

With this new technical lemma not only do we have a better picture of what maps lie in $\mathscr{D}_e^{(2)}$ but we also find another technical result that will be useful in our proof of Theorem 2.3.4. Like the previous lemma, this new corollary helps us to determine which maps are in $\mathscr{D}_e^{(2)}$ by showing which maps generate $\mathscr{D}^{(2)}$.

Corollary 2.3.6 ([Smo20], Corollary 6.6)**.** *Let R be a toric ring. Then the maps $\sum_{a,a'} c_{a,a'} \pi_a \otimes \pi_{a'}$ is compatible with the diagonal if and only if, for each $d \in \frac{1}{p^e}\mathbb{Z}^n$, we have $\sum_{a+a'=d} c_{a,a'} \pi_a \otimes \pi_{a'}$ is compatible with the diagonal. Additionally, it follows that $\mathscr{D}^{(2)}(R)$ is generated over k by the maps $\pi_d \in \mathscr{D}^{(2)}$.*

With both of these technical results doing the heavy lifting we can then present Smolkin's proof of Theorem 2.3.4.

Proof of Theorem 2.3.4. We will present the proof for when $R = T$ and note that it is not much work to extend the proof for T to any toric ring R. To check if $\pi_d \in \mathscr{D}^{(2)}(R)$, by Corollary 2.3.6 we need for there to exist a map

$$\phi = \sum_{a,a'} c_{a,a'} \pi_a \otimes \pi_{a'} \in \mathrm{Hom}(F_*^e(T \otimes T), T \otimes T)$$

such that ϕ is compatible with I_Δ and such that the sum $\sum_{a+a'=d} c_{a,a'} \neq 0$. Rewriting this sum, we can then rewrite this condition as

$$\sum_{a+a'=d} c_{a,a'} = \sum_{[u] \in \frac{1}{p^e}\mathbb{Z}^n/\mathbb{Z}^n} \sum_{a \in [u]} c_{a,d-a} \neq 0.$$

In order for this condition to be satisfied there would then have to exist some $[u] \in \frac{1}{p^e}\mathbb{Z}^n/\mathbb{Z}^n$ such that $\sum_{a\in[u]} c_{a,d-a} \neq 0$. Since we also require that ϕ be compatible with I_Δ, we can then use Lemma 2.3.5 to see that $\sum_{a\in[u]} c_{a,d-a} \neq 0$ for all $[u] \in \frac{1}{p^e}\mathbb{Z}^n/\mathbb{Z}^n$ because

$$\sum_{a\in[u_1]} c_{a,d-a} = \sum_{b\in[u_2]} c_{b,d-b}$$

for all $[u_1], [u_2] \in \frac{1}{p^e}\mathbb{Z}^n/\mathbb{Z}^n$. Because we need $\sum_{a\in[u]} c_{a,d-a} \neq 0$ for all $[u] \in \frac{1}{p^e}\mathbb{Z}^n/\mathbb{Z}^n$, we know that there will always be some $a \in [u]$ such that $c_{a,d-a} \neq 0$. Since $\phi \in \mathrm{Hom}(F^e_*(T\otimes T), T\otimes T)$ we know by Proposition 2.3.3 that $a, d-a \in \mathrm{int}(P_T)$ because π_a and π_{d-a} are basis elements of $\mathrm{Hom}_R(F^e_*T, T)$. However, we can rewrite this condition as $a \in P_T \cap (d - P_T)$ where $a \in [u]$ for all equivalence classes $[u]$ and we obtain the desired condition on π_d.

Conversely, for any d such that each equivalence class $[u_i] \in \frac{1}{p^e}\mathbb{Z}^n/\mathbb{Z}^n$ has a representative a_i in the interior of $P_T \cap (d - P_T)$. Then it follows from Smolkin's proof of Corollary 2.3.6 that the map $\sum_i \pi_{a_i} \otimes \pi_{d-a_i}$ is a map compatible with the diagonal and its restriction to the diagonal is π_d. Thus we have shown that $\pi_d \in \mathscr{D}^{(2)}(T)$ where $a \in \frac{1}{p^e}\mathbb{Z}^n \cap \mathrm{int}(P_R)$ and the interior of $P_R \cap (a - P_R)$ contains a representative of each equivalence class in $\frac{1}{p^e}\mathbb{Z}^n/\mathbb{Z}^n$, completing our proof of Theorem 2.3.4 when $R = T$ as we set out to. $\square$

Chapter 3

It Was Just Cones All Along!

In the previous chapter, we defined the sets $\mathscr{D}^{(2)}(R)$ and saw how involved computing such sets can be for toric rings. In order to streamline this computation process, we will show that computing $\mathscr{D}^{(2)}(R)$ is equivalent to solving the geometric problem we stated in the introduction. With this link established, we will then present a new algorithmic method of computing $\mathscr{D}^{(2)}(R)$ whenever $R = K[\check{\sigma} \cap \mathbb{Z}^2]$ with σ a strongly convex rational cone in $\mathbb{R}^2$.

3.1 Results

Beginning with our reformulation of the sets $\mathscr{D}^{(2)}(R)$, let us first introduce some key definitions.

Definition 3.1.1. Let $P \subseteq \mathbb{R}^d$. Then for any $v \in \mathbb{Z}^d$ we call the set

$$v + P := \{v + x \,|\, x \in P\}$$

an integer translate of P. Sometimes we will also say that $v + P$ is an integer translate of P by v.

Definition 3.1.2. Let $P \subseteq \mathbb{R}^d$. Then we say P covers $\mathbb{R}^d$ by integer translates when

$$\bigcup_{v \in \mathbb{Z}^d} v + P = \mathbb{R}^d.$$

Similarly, we say P tiles $\mathbb{R}^d$ by integer translates if P covers $\mathbb{R}^d$ by integer translates and $\operatorname{int}(v + P) \cap \operatorname{int}(v' + P) = \emptyset$ implies that $v = v'$ for $v, v' \in \mathbb{Z}^d$.

We can define a new set $\mathcal{D}^{(2)}(R)$ which will be the set of points that solve our problem from the introduction.

Definition 3.1.3. For $R = K[\check{\sigma} \cap \mathbb{Z}^d]$ a toric ring, we say $v \in \mathcal{D}^{(2)}(R)$ if and only if for all $x \in \mathbb{R}^d$ there exists an integer translate x' of x such that $x' \in P_R \cap (v - P_R)$. As before, if R is understood from context, or unspecified, we will write $\mathcal{D}^{(2)}$ rather than $\mathcal{D}^{(2)}(R)$.

To see that this set is indeed the set points that solve our problem from the introduction, we then need the following result.

Proposition 3.1.4. *Let $R = K[\check{\sigma} \cap \mathbb{Z}^d]$ be a toric ring. Then $v \in \mathcal{D}^{(2)}$ if and only if integer translates of $P_R \cap (v - P_R)$ cover $\mathbb{R}^d$.*

Proof. First suppose that $v \in \mathcal{D}^{(2)}(R)$. Then we know from our definition of $\mathcal{D}^{(2)}$ that for any $x \in \mathbb{R}^d$ there exists an integer translate $x' = x + t$ of x such that $x' \in P_R \cap (v - P_R)$. We can then see that $x \in (x - x') + (P_R \cap (v - P_R))$, $x - x' \in \mathbb{Z}^d$, and thus that $\mathbb{R}^d$ can be covered by integer translates of $P_R \cap (v - P_R)$.

For the other direction assume that $P_R \cap (v - P_R)$ covers $\mathbb{R}^d$. Then we know that for any $x \in \mathbb{R}^d$ that there exists some $t \in \mathbb{Z}^d$ such that $x \in t + (P_R \cap (v - P_R))$. This then tells us that $x - t \in P_R \cap (v - P_R)$ and, because $x - t$ is an integer translate of x, we have shown that $v \in \mathcal{D}^{(2)}(R)$ as desired. $\square$

In addition to providing context for our new $\mathcal{D}^{(2)}(R)$ definition, this interpretation of our $\mathcal{D}^{(2)}(R)$ allows us to see that $\mathcal{D}^{(2)} = \mathscr{D}^{(2)}$ are equal.

Corollary 3.1.5. *Let σ be a strongly convex rational cone and $R = K[\check{\sigma} \cap \mathbb{Z}^d]$. Then*

$$\mathscr{D}^{(2)}(R) = \mathcal{D}^{(2)}(R).$$

Proof. By Theorem 2.3.4 we know that the maps π_v that generate $\mathscr{D}^{(2)}(R)$ are precisely the maps such that $P_R \cap (v - P_R)$ contain representatives of each equivalence class in $\frac{1}{p^e}\mathbb{Z}^d/\mathbb{Z}^d$ for all e. This condition of containing each equivalence class for all e is then equivalent to $P_R \cap (v - P_R)$ covering $\mathbb{R}^2$ by integer translates. Thus it follows that $\mathscr{D}^{(2)}(R) = \mathcal{D}^{(2)}(R)$ as desired. $\square$

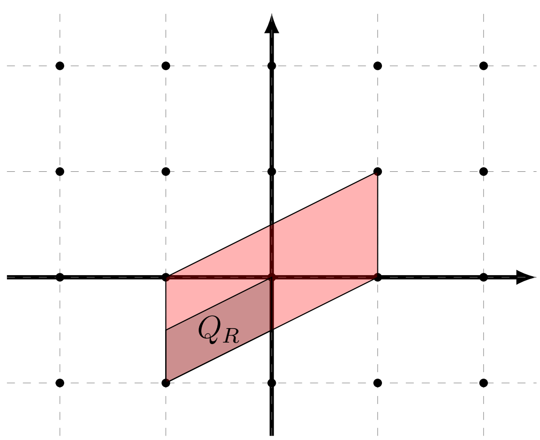

Figure 3.1: $P_R \cap -P_R$ in red along with the fundamental parallelogram Q_R in gray for the cone $\sigma(1, 2)$.

Since by proposition 3.1.5 our notation for $\mathcal{D}^{(2)}$ is interchangeable, from now on we will favor the $\mathcal{D}^{(2)}$ notation over the $\mathscr{D}^{(2)}$ notation to indicate we are using our new reformulation.

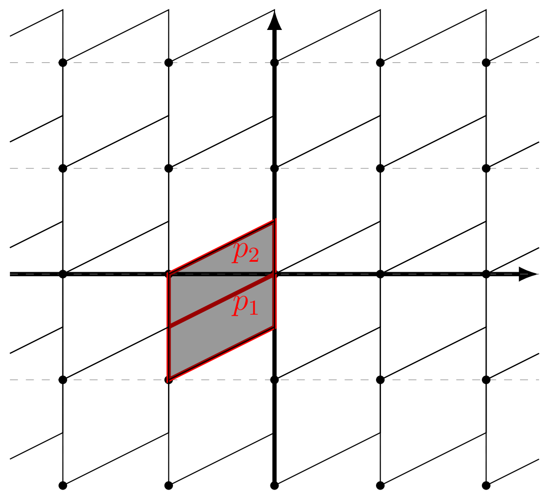

Figure 3.2: P' (grey) tiling $\mathbb{R}^2$ and the associated p_i outlined in red for the cone $\sigma(1,2)$.

From here, our next goal will be to compute $\mathcal{D}^{(2)}(R)$ toric rings of the form $R = K[\check{\sigma} \cap \mathbb{Z}^2]$ with σ a cone of the form $\sigma(\{e_1, -ae_1 + be_2\})$. Since we will only consider cones of the aforementioned form, it will be convenient to introduce the following definition.

Definition 3.1.6. Consider a cone of the form $\sigma(\{e_1, -ae_1 + be_2\})$. Then we will write $\sigma(a,b)$ to denote the cone $\sigma(\{e_1, -ae_1 + be_2\})$.

Definition 3.1.7. Let $R = K[\check{\sigma} \cap \mathbb{Z}^2]$ be a toric ring. Then the parallelogram

$$Q_R = P_R \cap \left(\left(-1, -\frac{a+1}{b}\right) - P_R\right)$$

is the fundamental parallelogram of R.

With this, we can proceed by first finding an obvious choice for elements of $\mathcal{D}^{(2)}$. We can see that the parallelogram $P' = P_R \cap \left((-1, \frac{b-a}{b}) - P_R\right)$ tiles $\mathbb{R}^2$ by integer translates since it has height 1 and width 1. This tells us then that $(-1, \frac{b-a}{b}) \in \mathcal{D}^{(2)}(R)$.

To find a bound for $\mathcal{D}^{(2)}$ we will notice that we can build the previous tiling parallelogram P' by stacking b translations of our fundamental parallelogram of R. If we label each of these translations of our fundamental parallelogram as $p_1, p_2, \ldots, p_b$ with $p_1 = Q_R$ and each successive p_i taking the form $p_i = (0, 1/b) + p_{i-1}$.

In Figure 3.2, we see the tiling parallelogram P' and that P' can be built by stacking p_2 on top of p_1. Since P' tiles $\mathbb{R}^2$ by integer translates, we then get the following lemma.

Lemma 3.1.8. *Let $v \in \mathbb{R}^2$ and let $P_v = P_R \cap v - P_R$. Then P_v covers $\mathbb{R}^2$ if and only if there exists an integer translate of p_i contained in P_v for all p_i. In other words, $v \in \mathcal{D}^{(2)}(R)$ if and only if there exists some $v_i \in \mathbb{Z}^2$ such that $v_i + p_i \subseteq P_v$ for $1 \leq i \leq b$.*

Proof. For the forward direction, suppose that there exist $v_i \in \mathbb{Z}^2$ such that $v_i + p_i \subseteq P_v$ for $1 \leq i \leq b$. Because $P' = \bigcup_{i=1}^{b} p_i$ and P' tiles $\mathbb{R}^2$ by integer translates, we can see that P_v covers $\mathbb{R}^2$ by integer translates because integer translates of P_v cover P'.

For the other direction, now assume that P_v covers $\mathbb{R}^2$. Since P_v covers $\mathbb{R}^2$, we know that for all $x \in p_i$ that there exists a point $v_x \in \mathbb{Z}^2$ such that $v_x + x \in P_v$. Since P_v is compact, it will then suffice to show that there exists some $v' \in \mathbb{Z}^2$ such that $v' + \operatorname{int}(p_i) \subseteq P_v$ since the compactness of P_v would then imply that $v' + p_i \subseteq P_v$. To see this, first observe that if $x = (x_1, x_2) \in \operatorname{int}(p_i)$ and $v_x + x \in P_v$, then for all $y \in p_x := \{(y_1, y_2) \in p_i | y_1 \leq x_1, ay_1 + bz_2 \leq -ax_1 + bx_2\}$ we can see that $v_x + y \in P_v$. This follows by considering the inequalities that bound P_v and noticing that if $v_x + x \in P_v$, then our conditions upon y by imposing that $y \in p_x$ tell us that $v_x + y$ satisfy the same inequalities needed to ensure that $v_x + y \in P_v$.

With this fact, we can proceed by contradiction and assume there does not exist a $v' \in \mathbb{Z}^2$ such that $v' + \operatorname{int}(p_i) \subseteq P_v$. Take some point $x \in \operatorname{int}(p_i)$ and consider the line segment ℓ_x that connects x to the upper right most vertex of p_i, which we will denote w. Since there does not exist a $v' \in \mathbb{Z}^2$ such that $v' + \operatorname{int}(p_i) \subseteq P_v$, we know that for any $v_x \in \mathbb{Z}^2$ such that $v_x + x \in P_v$ there must exist a point $z_1 \in \ell_x \setminus \{w\}$ such that $v_x + z \notin P_v$. By our previous observation, this tells us that

$$\{v_{z_1} \in \mathbb{Z}^2 \,|\, v_{z_1} + z_1 \in P_v\} \subset \{v_x \in \mathbb{Z}^2 \,|\, v_x + x \in P_v\}.$$

We can then iterate this process with $z_j \in \ell_{z_{j-1}} \setminus \{w\}$ to get the series of proper inclusions

$$\{v_x \in \mathbb{Z}^2 \,|\, v_x + x \in P_v\} \supset \{v_{z_1} \in \mathbb{Z}^2 \,|\, v_{z_1} + z_1 \in P_v\} \supset \cdots \supset \{v_{z_j} \in \mathbb{Z}^2 \,|\, v_{z_j} + z_j \in P_v\}$$

Since $\{v_x \in \mathbb{Z}^2 \,|\, v_x + x \in P_v\}$ is a finite set and P_v is bounded, this means that eventually there must exist some $z_j \in \operatorname{int}(p_i)$ such that $\{v_{z_j} \in \mathbb{Z}^2 \,|\, v_{z_j} + z_j \in P_v\} = \emptyset$. This then contradicts our assumption that P_v covers and thus we have shown there exists some $v' \in \mathbb{Z}^2$ such that $v' + \operatorname{int}(p_i) \subseteq P_v$. □

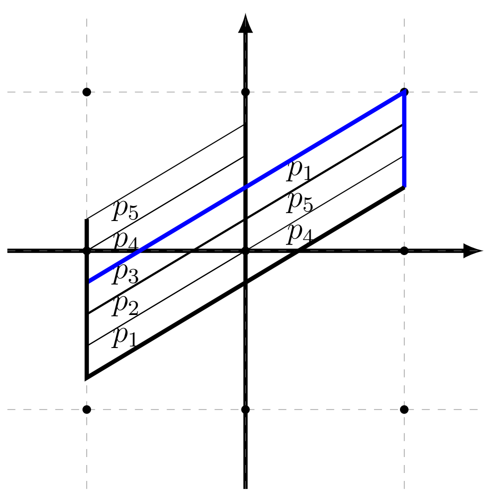

Figure 3.3: Minimal P_v (in blue) of width 2 containing integer translates of all p_i for $\sigma(3, 5)$

Corollary 3.1.9. *$v \in \mathcal{D}^{(2)}(R)$ if and only if P_v contains integer translates of all p_i.*

Proof. This is a restatement of Lemma 3.1.8. □

To see how this new formulation $\mathcal{D}^{(2)}(R)$ can help us compute $\mathscr{D}^{(2)}(R)$ let us consider the following example.

Example 3.1.10. Let us consider the cone $\sigma(\{e_1, -3e_1+5e_2\})$ and $R = K[\check{\sigma} \cap \mathbb{Z}^2]$ be the associated toric ring. To compute $\mathcal{D}^{(2)}(R)$, as we have shown in Corollary 3.1.9 we only need to care about the fundamental parallelogram Q_R and the translations of Q_R, $p_i = (0, \frac{i-1}{5}) + Q_R$ for $1 \leq i \leq 5$. Noting that $Q_R = p_1$, we can then see that $\bigcup_{i=1}^{5} p_i$ tiles $\mathbb{R}^2$ since it is a parallelogram with a height and width of 1. With these p_i established, we can then try to find the values of v such that P_v minimally covers an integer translation of the p_i. While we have seen already that the parallelogram $\bigcup_{i=1}^{5} p_i$ covers all p_i minimally, we can also find the other points by simply looking at the correct picture. For the parallelogram with width 2, to find the correct shift by v we can see in Figure that our minimal parallelogram is made up of 2 stacks of 3 p_i. Thus we get that the correct $v = (0, 1/5)$.

Overlaying all the minimal parallelograms onto one graph, as we can see is done in Figure 3.4, we can then see that the corners of each colored edge in Figure 3.1.10

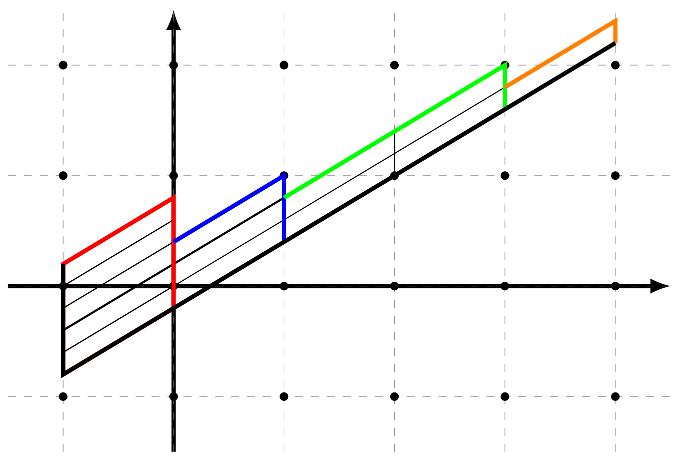

Figure 3.4: Stacks of p_i for Example 3.1.10

From Example 3.1.10 and Figure 3.4, we can start to see that the important points we need to find for computing $\mathcal{D}^{(2)}(R)$ are precisely the v such that we minimally contain translates of each p_i.

Definition 3.1.11. Let $R = k[\check{\sigma} \cap \mathbb{Z}^2]$ with $\sigma = \sigma(a, b)$. Then we let $v_i \in \mathbb{R}^2$ denote the point such that P_{v_i} is a parallelogram of width i and minimally covers translates of all p_j.

As we saw in 3.1.10, to find these v_i points it will suffice to find the minimal height of the parallelogram in terms of p_i. To do this we can note that v_1 will always yield the parallelogram $P_{v_1} = P'$ which is made by stacking b-many p_i in one column. To find P_{v_2}, we can add another column of p_i to P_{v_1} and remove rows of p_i until we no longer translates of all p_i. Stopping before the step before we no longer contain all p_i will then tell us P_{v_2}. We can then iterate this process, going from P_{v_j} to $P_{v_{j+1}}$, to find all v_j.

Now that we can compute these v_j reliably, all that is left is to show that finding these v_j points is sufficient to compute $\mathcal{D}^{(2)}(R)$. To show this, we begin with the following proposition.

Proposition 3.1.12. *Suppose that $v \in \mathcal{D}^{(2)}(R)$. Then*

$$v + \check{\sigma} \subseteq \mathcal{D}^{(2)}(R).$$

Proof. First note that $v \in \mathbb{R}$, and $v' \in C^\vee$,

$$v - P_R \subseteq (v + v') - P_R.$$

Then because $v \in \mathcal{D}^{(2)}(R)$ we know that $P_R \cap (v - P_R)$ covers $\mathbb{R}^2$ by integer translates. Since $(v + v') - P_R \supseteq v - P_R$, we can then see that

$$P_R \cap (v - P_R) \subseteq P_R \cap ((v + v') - P_R)$$

and thus that $P_R \cap ((v + v') - P_R)$ covers $\mathbb{R}^2$ by integer translates. By Proposition 3.1.5, it then follows that $v + v' \in \mathcal{D}^{(2)}$ as desired. □

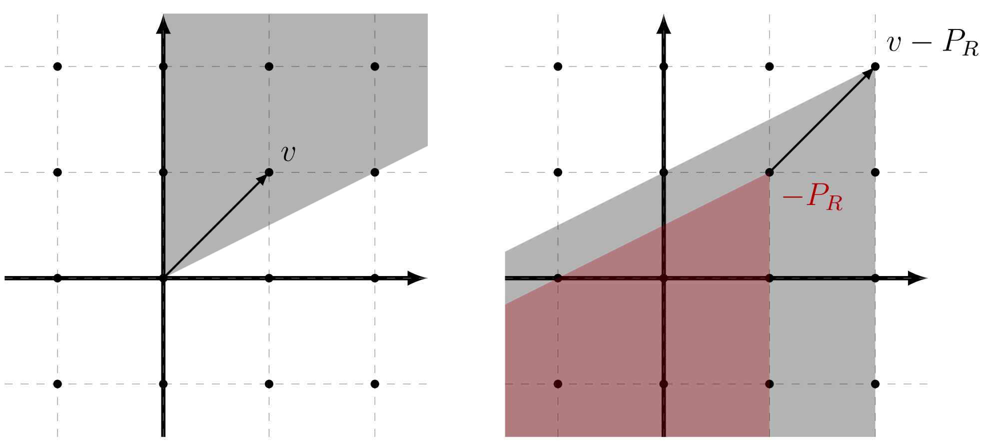

Figure 3.5: A geometric interpretation of the proof of Proposition 3.1.12: $\check{\sigma}$ and v graphed (left), $v - P_R$ on the right

This then tells us that $\bigcup_{i=1}^{b} v_i + \check{\sigma} \subseteq \mathcal{D}^{(2)}(R)$ and establishes $\bigcup_{i=1}^{b} v_i + \check{\sigma}$ as a lower bound for $\mathcal{D}^{(2)}(R)$. From here, we will then show that $\mathcal{D}^{(2)}(R) \subseteq \bigcup_{i=1}^{b} v_i + \check{\sigma}$ and we will have computed $\mathcal{D}^{(2)}(R)$. To do so, first write v_i as the coordinate pair $v_i = (x_i, y_i)$.

Then will show that for all $\varepsilon > 0$ and $1 \leq i \leq b$, $w_i = (x_i - \varepsilon, y_i - \varepsilon) \notin \mathcal{D}^{(2)}(R)$ and $w_i' = (x_{i+1} - \varepsilon, y_i + \frac{a}{b} - \varepsilon) \notin \mathcal{D}^{(2)}(R)$. Since these w_i and w_i' points make up the points just outside of the cones $v_i + \check{\sigma}$ and their intersections, showing that these points do not lie in $\mathcal{D}^{(2)}(R)$ will then prove that $\mathcal{D}^{(2)}(R) \subseteq \bigcup_{i=1}^{b} v_i + \check{\sigma}$.

Theorem 3.1.13. *Let σ be a strongly convex rational cone and $R = K[\check{\sigma} \cap \mathbb{Z}^2]$. Then*

$$\mathcal{D}^{(2)}(R) = \bigcup_{i=1}^{b} v_i + \check{\sigma}$$

Proof. First we are going to note that to show w_i and w_i' are not in $\mathcal{D}^{(2)}(R)$ it is sufficient to show that P_{w_i} and $P_{w_i'}$ do not cover $\mathbb{R}^2$ by integer translates respectively. For P_{w_i}, this is immediately clear because $P_{w_i} \subset P_{v_i}$ and we have already defined our v_i to be the points that minimally cover translates of each p_i.

For the $P_{w_i'}$, let us again let $v_i = (x_i, y_i)$. Then by definition we know these $P_{(x_i,y_i)}$ are parallelograms with i many columns of stacked p_i. We also know that the height of these columns must either stay the same of decrease due to our minimality assumption. From this we can notice that $P_{(x_{i+1}, y_i + \frac{a}{b})}$ is the parallelogram whose columns of p_j are the same height as the parallelogram $P_{(x_i,y_i)}$ but who has one more column. If $y_{i+1} = y_i + \frac{a}{b}$, then we are back in the w_i case and we are done. Otherwise, we know that $y_{i+1} \leq y_i + \frac{a}{b}$ and the parallelogram $P_{(x_{i+1},y_{i+1})}$ has shorter columns than $P_{(x_i,y_i)}$. Since our columns are shorter, our minimality assumption gives us that there must be some unique p_j in the right most column of $P_{(x_{i+1},y_{i+1})}$ that was uniquely found in the top most row of $P_{(x_i,y_i)}$. From here we then get that within $P_{(x_{i+1}, y_i + \frac{a}{b})}$ that the p_j from earlier only lies within the top row or the rightmost column of $P_{(x_{i+1}, y_i + \frac{a}{b})}$. This then tells us that p_j is not covered by $P_{w_i'}$ and it follows from Corollary 3.1.9 that $P_{w_i'} \notin \mathcal{D}^{(2)}(R)$. Thus we have shown that $\mathcal{D}^{(2)}(R) = \bigcup_{i=1}^{b} (x_i, y_i) + \check{\sigma}$. $\square$

Having now proven Theorem 3.1.13, we can see how helpful these v_i can be in computing $\mathcal{D}^{(2)}$. For instance, we can use Figure 3.4 to compute $\mathcal{D}^{(2)}(R)$ for the ring R from Example 3.1.10.

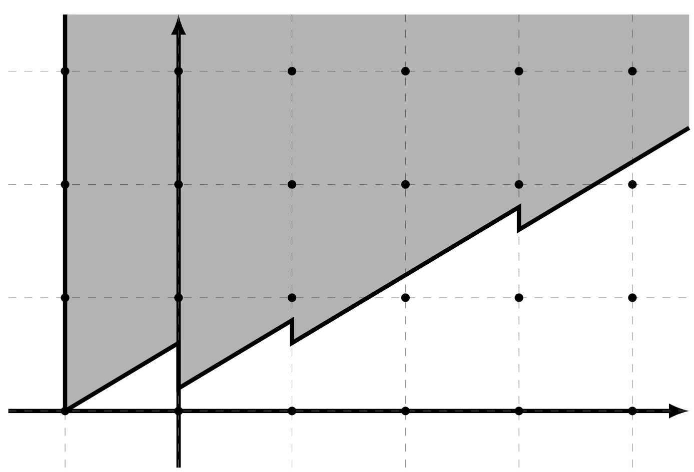

Figure 3.6: $\mathcal{D}^{(2)}(R)$ for $\sigma = \sigma(\{e_1, -3e_1 + 5e_2\})$

Additionally, since we now only care about the height of the columns in each P_{v_i}, we can see in the following example another way to determine the height of each P_{v_i}.

Example 3.1.14. To see another method in for finding the v_i of $\mathcal{D}^{(2)}(R)$ for a specific σ, let us consider the cone $\sigma(\{e_1, -3e_1 + 5e_2\})$. As we saw in the previous example, all that matters for finding $\mathcal{D}^{(2)}(R)$ is just what our minimal polynomials look like. However, instead of looking at the geometric pictures to compute $\mathcal{D}^{(2)}(R)$, we can use modular arithmetic in the group $\mathbb{Z}/5\mathbb{Z}$ to find out what the height of our minimal parallelograms must be. With this height, we will then have everything we need to compute $\mathcal{D}^{(2)}(R)$.

In order to use a modular arithmetic approach, we first need to figure out why $\mathbb{Z}/5\mathbb{Z}$ relates to the height of our parallelograms. First, since our minimal parallelograms are made up of columns of p_i stacked on each other, we can see that the bottom p_i in the first columns will be 1 and will increase in index by 3 for every column. However, since there are only 5 distinct p_i, eventually we will need to consider the columns modulo 5, thus highlighting our reason for working modulo 5. Considering our problem in this way, a column of stacked j-many stacked p_i corresponds to a set of j-many consecutive elements of $\mathbb{Z}/5\mathbb{Z}$. Similarly, the starting point of each of these sets will be determined by the first p_i in the column, and thus we know each of these sets will have starting points staggered by 3 modulo 5. To find the height of a minimal parallelogram of width m, the problem becomes computing the minimal size our staggered sets must be in order to completely cover $\mathbb{Z}/5\mathbb{Z}$. To find this size, we can use another helpful picture. As seen in Figure 3.7, at each step i in the splitting of our blocks we see that we recover the heights of minimally covering polynomials pictured in Figure 3.7 by simply finding the size of the largest clump of blocks.

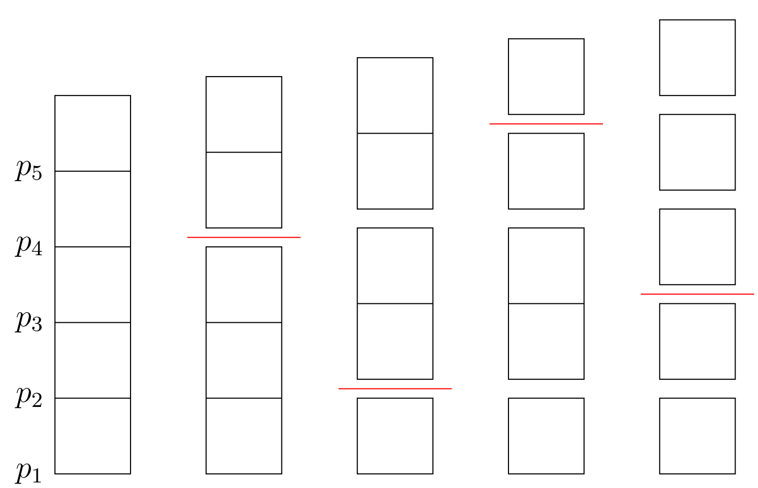

Figure 3.7: Visual representation of Example 3.1.14

Remark 3.1.15. The method established in Example 3.1.14 only works because we are considering cones of the form $\sigma(\{e_1, -ae_1 + be_2\})$ where a and b are relatively prime. If a and b were not relatively prime eventually you could wind up with clumps of more than 1 block that will never be split which is an issue.

3.2 Future Questions

While these examples, along with the rest of the work in this chapter, showcase an easy and workable method of computing $\mathcal{D}^{(2)}(R)$ for all toric rings arising from two-dimensional cones, there are a variety of interesting questions out there to be solved. Most pertinent to the work we have presented, it is not clear that there exists an explicit form for the set $\mathcal{D}^{(2)}(R_\sigma)$ that depends entirely on σ and its generating set and determining whether or not such a form exists could be an interesting combinatorial problem attempting to leverage our methods of computing $\mathcal{D}^{(2)}(R_\sigma)$ in the examples.

Another interesting question that arises is whether or not this method can be generalized to computing $\mathcal{D}^{(2)}(R_\sigma)$ for higher dimensional σ. In two dimensions our method works because of the convenient geometry of parallelograms whereas in higher dimensions the complex geometry of P_R could pose issues. While it is likely that this method of finding some minimal p_i that $P_R \cap v - P_R$ may not behave as nicely geometrically, it would interesting to see whether or not a refinement in the choice of p_i could lead to a nice bound on $\mathcal{D}^{(2)}(R_\sigma)$.

The last area we will touch on for how this thesis could be expanded on revolves around a class of Cartier algebras we did not touch on in the thesis.

Definition 3.2.1. We define the nth diagonal Cartier algebra on R to be

$$\mathscr{D}^{(n)}(R) := \mathscr{C}^{R^{\otimes n}, I_\Delta \circlearrowleft}|_{R^{\otimes n}/I_\Delta}$$

where $R^{\otimes n}$ is the k-tensor product of n copies of R.

Just like $\mathscr{D}^{(2)}(R)$ has a nice geometric interpretation when R is a toric ring, the problem of computing $\mathscr{D}^{(n)}(R)$ also can be made geometric when R is a toric ring. Similarly to $\mathscr{D}^{(2)}(R)$, $\mathscr{D}^{(n)}(R)$ can also be re-characterized geometrically.

Theorem 3.2.2 ([PST18], Theorem 3.4)**.** *For $R = K[\check{\sigma} \cap \mathbb{Z}^d]$ a toric ring, we say $v \in \mathscr{D}^{(n)}(R)$ if and only if for all $v_1, \ldots, v_{n+1} \in \mathbb{R}^d$ there exist integer translates $v_1', \ldots, v_{n+1}'$ of the original v_i such that $v_i' \in P_R$ and $\sum_{i=1}^{n+1} v_i' \in (v - P_R)$.*

Although computing these $\mathscr{D}^{(n)}(R)$ is a difficult problem even in two dimensions, a natural extension of this thesis would be to attempt to compute such $\mathcal{D}^{(n)}(R)$. In fact, one such place to start would be attempting to find some lower bound to the set $\mathcal{D}^{(n)}(R)$ leveraging our newfound understanding of $\mathcal{D}^{(2)}(R)$.

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