General Session on Algebra: Searching for Toric Rings with USTP

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Ordinary Power of an Ideal

Let R be a Noetherian ring.

Definition

Let $\mathfrak{p}$ be an ideal of R. The n^{th} **ordinary power** of $\mathfrak{p}$, denoted $\mathfrak{p}^n$, is the ideal generated by all products of n elements of $\mathfrak{p}$.

Remark

When $i > j$, we have $\mathfrak{p}^i \subseteq \mathfrak{p}^j$.

Example

Let $R := k[x, y]$ and $\mathfrak{p} := (x, y)$.

Then $\mathfrak{p}^2 = (x^2, xy, y^2)$ and $\mathfrak{p}^3 = (x^3, xy^2, x^2y, y^3)$.

Symbolic Power of a Prime Ideal

Let R be a Noetherian ring.

Definition

Let $\mathfrak{p}$ be a prime ideal of R. The n^{th} **symbolic power** of $\mathfrak{p}$, denoted $\mathfrak{p}^{(n)}$, is the ideal

$$\mathfrak{p}^{(n)} := \{x \in R \mid \exists\ s \in (R \setminus \mathfrak{p}), xs \in \mathfrak{p}^n\}.$$

Remark

- When $i > j$, we have $\mathfrak{p}^{(i)} \subseteq \mathfrak{p}^{(j)}$.
- For fixed i, we have $\mathfrak{p}^i \subseteq \mathfrak{p}^{(i)}$.

Symbolic Power of a Prime Ideal (continued)

Example

Let $R := k[x, y]$ and $\mathfrak{p} := (x, y)$.

Then $\mathfrak{p}^{(2)} = (x^2, xy, y^2)$ and $\mathfrak{p}^{(3)} = (x^3, xy^2, x^2y, y^3)$.

Example

Let $R = \frac{k[x,y,z]}{(xy-z^2)}$ and $\mathfrak{p} := (x, z)$.

Then $\mathfrak{p}^{(2)} = (x) \supsetneq (x^2, xz, z^2) = \mathfrak{p}^2$.

Uniform Symbolic Topology Property (USTP)

Question

How far from equality does the containment $\mathfrak{p}^n \subseteq \mathfrak{p}^{(n)}$ lie across the ideals of some ring R?

Definition

A ring R is said to have the **Uniform Symbolic Topology Property** if there exists h such that for all prime ideals $\mathfrak{p}$ of R and all $n > 0$,

$$\mathfrak{p}^{(hn)} \subseteq \mathfrak{p}^n.$$

In some sense, this says that the "difference" between $\mathfrak{p}^{(n)}$ and $\mathfrak{p}^n$ varies *uniformly* among all the different ideals $\mathfrak{p} \subset R$, where that uniform difference is captured by the value h.

Toric Rings

Let $v_i \in \mathbb{Z}^n$ be a finite collection of vectors and k a field.

Definition

Let $R = k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. Then the **toric ring** associated to the vectors v_i is the subring of R generated by all monomials $x_1^{\lambda_1} \cdots x_n^{\lambda_n} = x^\lambda$ for which $\langle \lambda, v_i \rangle \geq 0$ for all i.

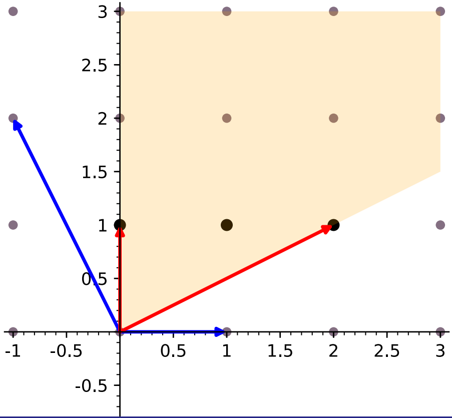
Example

For $(1,0)$ and $(0,1)$, the associated toric ring is $k[x,y]$.

Toric Rings (continued)

Example

For $(1,0)$ and $(-1,2)$, the associated toric ring is $k[y, xy, x^2y]$.



The Sets

Let R be a toric ring associated to the vectors v_i. Work in [Smo18] and [CS18] uses the Frobenius endomorphism of the R to define a set $\mathscr{D}^{(m)}$ which detects when R has USTP.

Specifically, they show that if these sets $\mathscr{D}^{(m)}$ are "sufficiently large" for all $m \geq 2$, then R has USTP.

Definition

We say that $\mathscr{D}^{(m)}$ is **sufficiently large** if $\mathscr{D}^{(m)}$ contains an epsilon ball centered around the origin.

Results in Two Dimensions

Using work from [CS18], we show

Proposition ([J19])

The only two-dimensional toric ring for which $\mathscr{D}^{(2)}$ is sufficiently large is the toric ring associated to e_1 and e_2.

Remark

Let R be the toric ring associated to e_1 and e_2. Then $\mathscr{D}^{(2)} = \mathscr{D}^{(m)}, m \geq 3$. Thus, the sets $\mathscr{D}^{(m)}$ are sufficiently large for all m, so R has USTP.

Results in Three Dimensions

Using work from [CHP+16], we show

Proposition ([J19])

There are only 2 three-dimensional toric rings for which $\mathscr{D}^{(2)}$ *is sufficiently large: the toric ring associated to* e_1*,* e_2*, and* e_3 *and the toric ring associated to* e_1*,* e_2*,* e_3*, and* $(-1, 1, 1)$*.*

Proposition ([CS18;J19])

In the case of R *associated to* e_1*,* e_2*,* e_3*, and* $(-1, 1, 1)$*, the sets* $\mathscr{D}^{(m)}$ *are sufficiently large for all* m*, so* R *has USTP.*

Remark

In the case of R associated to e_1, e_2, and e_3, we have $\mathscr{D}^{(2)} = \mathscr{D}^{(m)}, m \geq 3.$ Thus, the sets $\mathscr{D}^{(m)}$ are sufficiently large for all m, so R has USTP.

Results in Higher Dimensions

Let R be the toric ring associated to the vectors

$$\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\-1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\-1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\-1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\0\\-1\\1\end{pmatrix}.$$

By work in [PST18], the set $\mathscr{D}^{(2)}$ is sufficiently large.

Proposition ([J19])

The set $\mathscr{D}^{(3)}$ *is sufficiently large. Thus, there exists* $h \leq \dim R = 5$ *such that for all prime ideals* $\mathfrak{p}$ *of* R,

$$\mathfrak{p}^{(3h)} \subset \mathfrak{p}^3.$$

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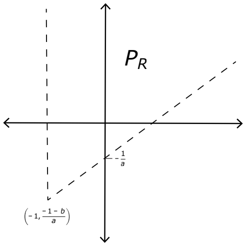
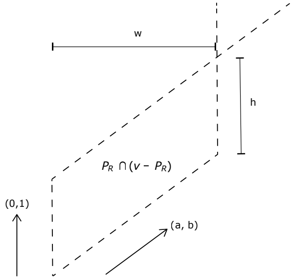
Working with $\mathscr{D}^{(2)}$ in $\mathbb{R}^2$

Thus, to understand $\mathscr{D}^{(2)}$ we need to understand when $P_R \cap (v - P_R)$ tiles $\mathbb{R}^2$ by integer translations.

Remark

In two-dimensions, the intersections $P_R \cap (v - P_R)$ are all parallelograms.





Tiling in $\mathbb{R}^3$

To show that $P_R \cap (v - P_R)$ can tile $\mathbb{R}^3$, we consider the cross sections of level planes. These cross sections are 2-dimensional.

We can prove that this polytope tiles $\mathbb{R}^3$ by showing that all cross sections of level planes in some unit interval – say, from $-\frac{1}{2}$ to $\frac{1}{2}$ – can tile $\mathbb{R}^2$.