Exploring a Connection Between Abstract Algebra and Geometry

Dylan Johnson

Advisors: Karl Schwede, Daniel Smolkin, Marcus Robinson

The University of Utah

Two Important Players

This project investigates an important interaction between two famous mathematical players: algebra and geometry. In particular, we hope to use geometry — especially questions of "tiling" — to understand the behaviour of certain algebraic functions.

How They Work Together

In [Smo18] and [CS18], J. Carvajal-Rojas and D. Smolkin developed a way of studying the zeroes of certain polynomials using geometry (and combinatorics). They let R denote some special set of polynomials and define a related set of points, called P_R.

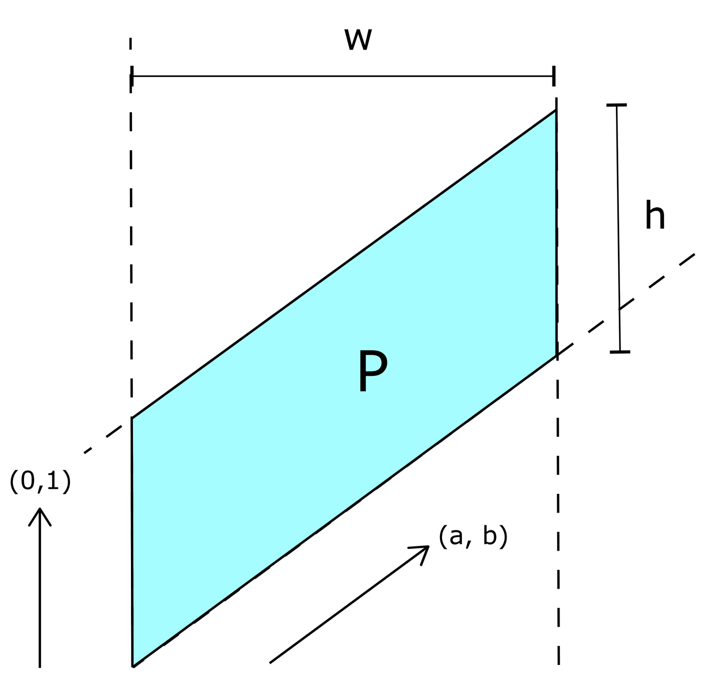

Figure 1: The parallelogram $P = P_R \cap (-P_R)$

Then, they show that the intersection of P_R and $-P_R$ (Figure 1) detects certain properties of the polynomials in R. This amounts to determining if the parallelogram P tiles the plane by unit translations up, down, left, and right. When it does tile, we obtain information about R. Hence, this project has investigated such questions of tiling to ultimately gain insight into the zeroes of certain polynomials.

Some Example Tilings

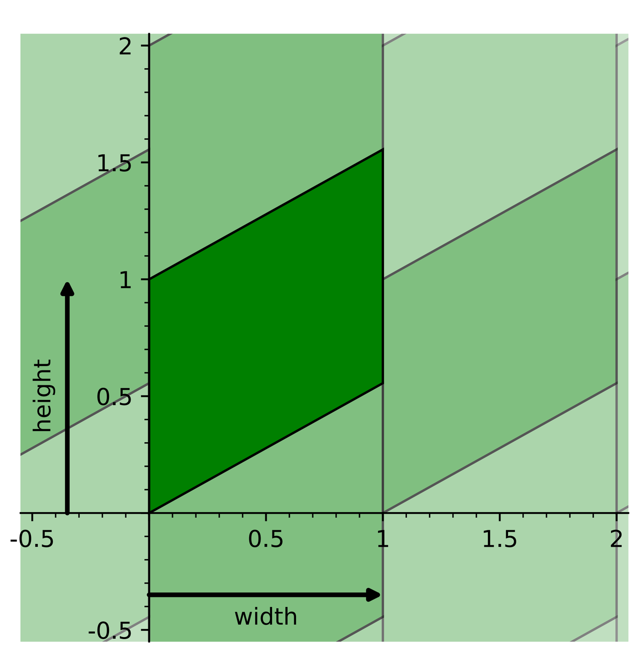

Fig. 1: A parallelogram of width 1 and height 1 which does tile.

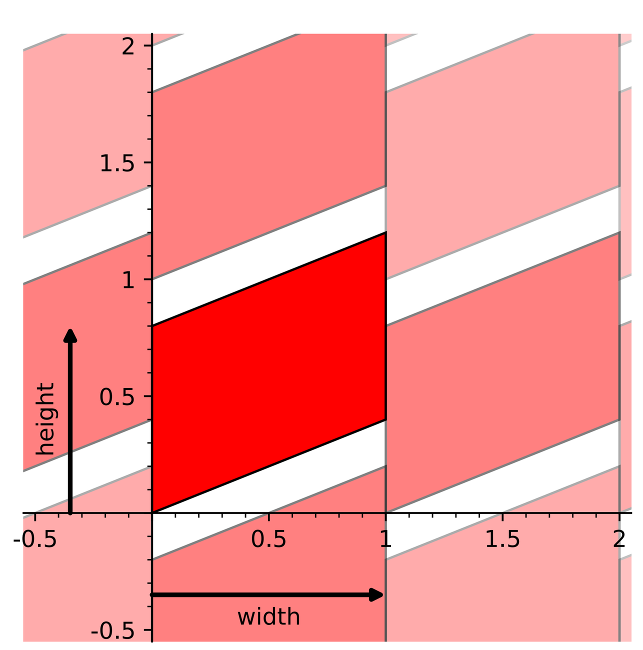

Fig. 2: A parallelogram of width 1 and height .8 which does not tile.

Geometric Results

A parallelogram is uniquely defined by its height, its width, and the slope of its lower edge. In particular, if a parallelogram has slope $\frac{a}{b}$, we show that the smallest such parallelogram which tiles has an integer width (e.g. 1, 2, 3, etc.) and height that is a multiple of $\frac{1}{b}$ (e.g. $\frac{1}{b}$, $\frac{2}{b}$, $\frac{3}{b}$, etc.). Figure 3 depicts the strategy for proving these claims.

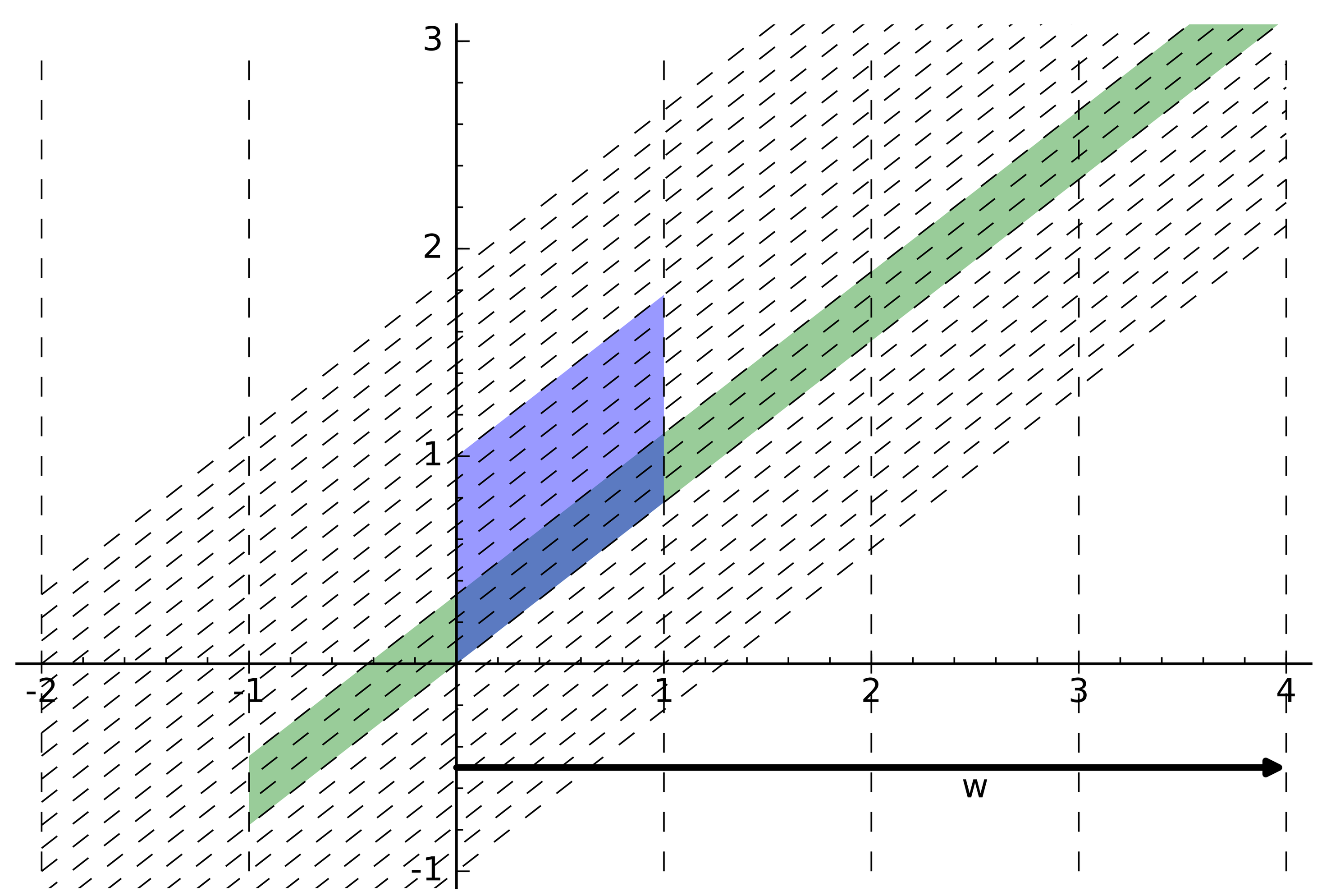

Fig. 3: A key step in reducing the case of an arbitrary parallelogram to one with specific width/height.

Using these observations, we conclude that the only parallelogram of interest has slope 0, height 1, and width 1: the unit square. Thus, we prove geometrically that the method developed by Carvajal-Rojas and Smolkin yields information – in two dimensions – about a single set of polynomials R, discussed next.

Algebraic Meaning

The corresponding R for the square with side lengths 1 is

$$R = \{\text{set of all polynomials in two variables } x \text{ and } y\};$$

polynomials in R look like $x + y^2$ or $3xy - xy^3$. Then, because the unit square tiles the plane, we have that for any curve ℓ, there exists $h \le 2$ such that the set

$$\{\text{polynomials in } R \text{ which are zero on } \ell \text{ of order at least } hn\}$$

is contained in the set

$$\{n^{th} \text{ powers of polynomials in } R \text{ which are zero on } \ell\}$$

for any n. In this case, $h = 1$. For more complicated examples where the polynomial sets have much interesting forms, we move to higher dimensions.

Higher Dimensions

So far, the poster has focused on two-dimensional tilings; we can in fact consider tilings in arbitrarily high dimensions. We use work from [Cho+16] to classify the three-dimensional cases which are of interest: there are only two, one of which the following figure depicts.

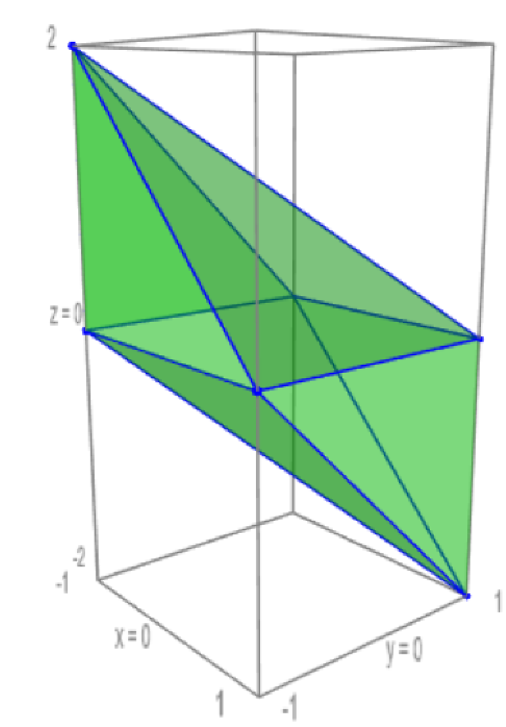

Fig. 4: A 3-dimensional example of $P_R \cap -P_R$.

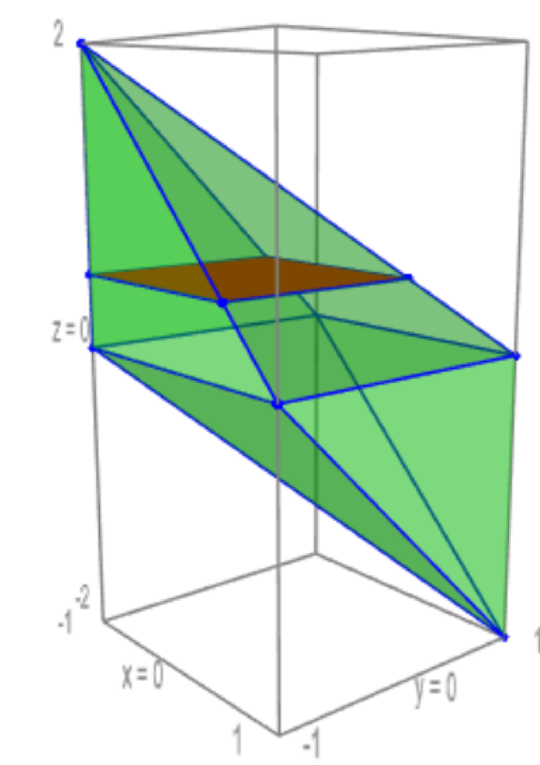

Fig. 5: The same 3-dimensional shape with a cross-section at $z = \frac{1}{2}$.

Our most recent work considers a set of polynomials R whose associated region $P_R \cap -P_R$ lives in five-dimensions, with 16 boundary planes. It is of a special type, called "Hibi", and was chosen because it has been previously studied in [PST18].

Acknowledgements

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References

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