Math 431 homework Due Friday, September 7, 2001

1. (Dummit and Foote) If x, y, z are real numbers such that

$$x + y + z = 1, \quad x^2 + y^2 + z^2 = 2, \quad x^3 + y^3 + z^3 = 3$$

show that for all $n \in \mathbf{Z}_{\geq 0}$, the number $x^n + y^n + z^n$ is rational. Find $x^4 + y^4 + z^4$. Show that x, y, z are not rational.

2. What is the dimension of the vector space of all homogeneous polynomials in x, y, z of total degree 5? What is the dimension of the subspace of symmetric polynomials? (i.e., polynomials unchanged under all permutations of the three variables)

3. Let e_i denote the i-th elementary symmetric function of n variables $x_1, \cdots, x_n$. (Implicitly, $e_i = 0$ for $i > n$.) Let

$$p_k = \sum_{i=1}^{n} x_i^k$$

be the k-th power sum of the variables. Verify Newton's identities: for all positive integers k,

$$\sum_{j=0}^{k-1} (-1)^j p_{k-j} e_j + \quad (-1)^k k e_k = 0.$$

Prove that the $p's$ are polynomials, with rational coefficients, in the $e's$, and vice versa.

4. Find, with proof, the following degrees:

- $[\mathbf{Q}(\sqrt{2}) : \mathbf{Q}]$
- $[\mathbf{Q}(\sqrt{2} + \sqrt{3}) : \mathbf{Q}]$
- $[\mathbf{Q}(\sqrt{3 + 4i}) : \mathbf{Q}]$, where, here, $i = \sqrt{-1}$
- $[\mathbf{Q}(\sqrt{3 + 2\sqrt{2}}) : \mathbf{Q}]$
- $[\mathbf{Q}(\sqrt{3 + 4i} + \sqrt{3 - 4i} : \mathbf{Q}]$, where, for the sake of definiteness, we take the first square root to be the one that lies in the first quadrant of the complex plane, and the second square root to the the one that lies in the fourth quadrant.