Math 431 homework Due Friday, September 21, 2001

For the remainder of this homework, let F be a quadratic field, i.e., a field of degree two over the rational numbers. We fix the following notation.

- d is a square-free, non-square, integer, and $F = \mathbf{Q}(\sqrt{d})$;
- $\mathbf{Z}_F = \mathbf{Z}[\delta]$, where $\delta = (1+\sqrt{d})/2$ if $d \equiv 1 \bmod 4$, and $\delta = \sqrt{d}$ otherwise;
- the result of applying the nontrivial automorphism of F to u is denoted u';
- the norm of $u \in F$ is $N(u) = uu'$, and the trace is $T(u) = u + u'$
- D, the "discriminant" of F, is $(\delta - \delta')^2$ which is equal to d or $4d$ according as to whether d is 1 mod 4 or not.
- F is an "imaginary" quadratic field if $d < 0$ and a "real" quadratic field if $d > 0$.

1. An element u of $\mathbf{Z}_F$ is a unit if and only if $N(u) = \pm 1$. A unit is nontrivial if it is not equal to ± 1. Find all nontrivial units in all imaginary quadratic fields. Find a nontrivial unit in $\mathbf{Q}(\sqrt{3})$. Find a nontrivial unit in $\mathbf{Q}(\sqrt{6})$. Find a nontrivial unit in $\mathbf{Q}(\sqrt{29})$.

2. Let $\omega = e^{2\pi i/3} = (-1 + \sqrt{-3})/2$. Show that $\mathbf{Z}[\omega]$ is a Euclidean domain (with respect to the function $N(a + b\omega) = a^2 - ab + b^2$.) Show 5 is a prime element in $\mathbf{Z}[\omega]$ but that 3 and 7 are not. What is the order of the quotient $\mathbf{Z}[\omega]/I$ where I is the principal ideal $(1 - \omega)$. Same question for $I = (5)$.

3. Let p be an odd prime. Prove that $(x^p-1)/(x-1) = x^{p-1}+x^{p-2}+\cdots+x+1$ is irreducible, so that $[\mathbf{Q}(e^{2\pi i/p}) : \mathbf{Q}] = p-1$. (Hint: consider $f(x+1)$.) Prove that $[\mathbf{Q}(\cos(2\pi/p) : \mathbf{Q}] = (p-1)/2$. (Hint: Find an automorphism that fixes the cosine.)

4. Let $F = \mathbf{Q}(\sqrt{2})$. Let $u = -1 + \sqrt{2}$. Prove that $\mathbf{Z}_F^* = \{\pm u^k : k \in \mathbf{Z}\}$. (By negating and reciprocating as necessary, we can take u to be a real number bigger than 1, and by considering $(u-u')^2 = b^2 D$ the number can be bounded away from 1.)

5. Find all integers that are simultaneously triangular and square, such as 1 and 36. (A little algebraic juggling will lead you to the equation $x^2-2y^2 = 1$.)

6. Suppose that K is an algebraic number field, and that α is integral over K in the sense that it satisfies a monic polynomial whose coefficients are in $\mathbf{Z}_K$. Prove that α is an algebraic integer.