Math 431 Halloween homework Due Friday, November 2, 2001

1. Show that if I and J are relatively prime ideals in a number ring and their product is a k-th power, then each is a k-th power. Show that if k is relatively prime to the order of the class group $\mathrm{Cl}(F)$, and I^k is principal, then I is principal.

From now on, let $F = \mathbf{Q}(\sqrt{-355})$.

2. How many ideals are there in $\mathbf{Z}_F$ of norm $75245625 = 3^2 \cdot 5^4 \cdot 7^3 \cdot 13$?

3. Find the class group of F.

4. Find all integer solutions to the equation

$$y^2 + 355 = x^3.$$

You may want to show that (a) in any solution x is odd, (b) consider the equation in principal ideals in $\mathbf{Z}_F$ that corresponds to a solution and show that the two ideals on the LHS are relatively prime, and (c) show that for any solution (x, y), the ideal $(y + \sqrt{-355})$ is the cube of a principal ideal.

5. Show that there is a prime ideal P such that

$$P = \mathrm{spn}_{\mathbf{Z}}(5, -3 + \delta).$$

(The index can be computed immediately, but you have to check that the RHS is an ideal.) Show that

$$N(5x + (-3 + \delta)y)/N(P) = 5x^2 - 5xy + 19y^2.$$

Show that a number n is represented by this quadratic form if and only if there is an ideal I of norm n that is in the ideal class $[P']$. Find a representation of $40131 = 3^2 \cdot 7^3 \cdot 13$ by this form.