Math 431 homework Due Wednesday, October 10, 2001

1. Find odd square-free integers a and b such that $(\sqrt{a}+\sqrt{b})/2$ is an algebraic integer.

2. Prove that $D(fg) = D(f)D(g)\mathrm{Res}(f,g)^2$ where $D(f)$ denotes the discriminant of a polynomial f.

3. Prove or disprove: If E and F are distinct cubic extensions of $\mathbf{Q}$, and K is a number field that contains both, then the degree of K over $\mathbf{Q}$ is divisible by 9.

4. Prove that the discriminant of the polynomial $f(x) = x^n + ax + b$ is

$$D(f) = (-1)^{n(n-1)/2} n^n b^(n-1) + (-1)^{(n+2)(n+3)/2}(n-1)^{n-1}a^n.$$

5. Show that $(1-\zeta_8)^4$ is an associate of 2 in $\mathbf{Z}[\zeta_8]$ where $\zeta_8 = e^{2\pi i/8}$ is a primitive eighth root of unity.

6. Show that if $\mathrm{m}_\alpha(x) = x^3 - x - 2x^2 - 8$ and $F = \mathbf{Q}(\alpha)$ then

$$(\alpha^2+\alpha)/2 \in \mathbf{Z}_F \backslash \mathbf{Z}[\alpha].$$

7. Let p be an odd prime, $\zeta = e^{2\pi i/p}$, and $F = \mathbf{Q}(\zeta)$. Find $N_{F/\mathbf{Q}}(\zeta)$. Find $N_{F/\mathbf{Q}}(\zeta - 1)$.

8. Let I be a nontrivial ideal in the ring of integers $\mathbf{Z}_F$ of a number field F. Prove that there is a nonzero element in $I \cap \mathbf{Z}$.

9. Find an example of nonunique factorization in $\mathbf{Q}(\sqrt{-15})$. I.e., find an element that factors into irreducibles in two genuinely different ways.