Math 431 homework Due Friday, October 26, 2001

1. Let $I = (2, 1+\sqrt{-3}) \subset \mathbf{Z}[\sqrt{-3}]$. Show that $I^2 = 2I$. Why does this not contradict unique factorization of ideals in a Dedekind domain?

2. Let $F = \mathbf{Q}(\alpha)$, where $\alpha^3 - \alpha^2 - 2\alpha - 8 = 0$. Show that $\mathbf{Z}_F = \mathrm{spn}_{\mathbf{Z}}(1, \alpha, (\alpha^2+\alpha)/2)$. Find three nontrivial ring homomorphisms from $\mathbf{Z}_F$ to $\mathbf{F}_2$.

3. Show that any Dedekind domain with finitely many prime ideals is a PID. (Hint: Use the CRT to find a generator of an ideal.)

4. Show that if I is any ideal in a number ring $\mathbf{Z}_F$ then $\mathbf{Z}_F/I$ is a PID.

5. Let $F = \mathbf{Q}(\alpha)$ where $\alpha^3 + \alpha + 1 = 0$. Show that in the ring $\mathbf{Z}_F$

$$(3) = PQ, \text{ where } P = (3, \alpha+2), Q = (3, \alpha^2+\alpha+2)$$

$$(31) = P^2Q, \text{ where } P = (31, \alpha-14), Q = (31, \alpha-3).$$