Math 431 Quiz Due Friday, October 12, 2001

Do any six of the following problems.

Eight hour time limit; hints available in L305.

1. Let α be a root of $f(x) = x^3 - 4x - 1$ and let $F = \mathbf{Q}(\alpha)$. Prove that $\mathbf{Z}_F = \mathbf{Z}[\alpha]$. Find a unit in $\mathbf{Z}_F$. Show that $2\mathbf{Z}_F$ is not a prime ideal (hint: to find two algebraic integers whose product is in the ideal, you could choose one of them to be $\alpha + 1$.) Find $\mathrm{Tr}_{F/\mathbf{Q}}(\alpha^k)$ for $k = 0, 1, 2, 3$.

2. Let $u_0, u_1, \cdots$ be a sequence of integers (i.e., elements of $\mathbf{Z}$) such that

$$u_{n+3} - 4u_{n+1} - u_n = 0$$

for all $n \geq 0$. Prove that there are numbers a_i so that

$$u_n = \sum_{i=1}^{3} a_i \left(\alpha^{(i)}\right)^n$$

where α is as given in the previous problem. (Hint: consider the first 3 terms and the *** matrix.) What can you say about

$$\lim_{n\to\infty} |u_n|^{1/n} \quad ?$$

3. Describe all solutions to the diophantine equation

$$x^2 + xy - 8y^2 = 1.$$

4. Show that if a polynomial with rational coefficients has r_1 real roots and r_2 pairs of complex conjugate roots, then the sign of its discriminant is $(-1)^{r_2}$. (Hint: One easy way to do this is to factor the polynomial over $\mathbf{R}$, and then use formulae.) Show that if F is a number field with r_1 real embeddings and r_2 pairs of complex conjugate embeddings then the sign of its discriminant is $(-1)^{r_2}$.

5. Show that if I is a nonzero ideal in the ring of integers $\mathbf{Z}_F$ of a number field then $I \cap \mathbf{Z}$ is a nonzero ideal in $\mathbf{Z}$.

6. For each prime ideal P in $\mathbf{Z}[i]$ (see the class notes for 9/21 for an explicit list) find, with explanation, the index $[\mathbf{Z}[i] : P]$ of P in $\mathbf{Z}[i]$.

7. Let α and β be algebraic integers of degree n that satisfy

$$\mathrm{Tr}_{\mathbf{Q}(\alpha)/\mathbf{Q}}(\alpha^k) = \mathrm{Tr}_{\mathbf{Q}(\beta)/\mathbf{Q}}(\beta^k)$$

for $0 \leq k \leq n$. Prove that α and β are conjugates.