THE FOURIER TRANSFORM AND THE MELLIN TRANSFORM

For suitable functions
$$f : \mathbb{R} \longrightarrow \mathbb{C}$$
the *Fourier transform of* f is the integral
$$\mathcal{F}f : \mathbb{R} \longrightarrow \mathbb{C}, \quad (\mathcal{F}f)(y) = \int_{\mathbb{R}} f(x) e^{-ixy}\,\mathrm{d}x,$$
and for suitable functions
$$g : \mathbb{R} \longrightarrow \mathbb{C}$$
the *inverse Fourier transform of* g is the integral
$$\mathcal{F}^{-1}g : \mathbb{R} \longrightarrow \mathbb{C}, \quad (\mathcal{F}^{-1}g)(x) = \frac{1}{2\pi} \int_{\mathbb{R}} g(y) e^{iyx}\,\mathrm{d}y.$$
The *Fourier inversion formula* says that if the functions f and g are well enough behaved then $g = \mathcal{F}f$ if and only if $f = \mathcal{F}^{-1}g$.

The exponential map is a topological isomorphism
$$\exp : (\mathbb{R}, +) \longrightarrow (\mathbb{R}^+, \cdot)$$
The Mellin transform, inverse Mellin transform, and Mellin inversion formula are essentially their Fourier counterparts passed through the isomorphism.

Specifically, given a suitable function on the positive real axis,
$$f : \mathbb{R}^+ \longrightarrow \mathbb{C},$$
we can make a corresponding function on the real line,
$$\widetilde{f} : \mathbb{R} \longrightarrow \mathbb{C}, \quad \widetilde{f} = f \circ \exp.$$
The Fourier transform of $\widetilde{f}$ is $\mathcal{F}\widetilde{f} : \mathbb{R} \longrightarrow \mathbb{C}$ where
$$\begin{aligned}
(\mathcal{F}\widetilde{f})(y) &= \int_{\mathbb{R}} \widetilde{f}(x) e^{-ixy}\,\mathrm{d}x \\
&= \int_{\mathbb{R}} f(e^x)(e^x)^{-iy}\,\frac{\mathrm{d}(e^x)}{e^x} \\
&= \int_{\mathbb{R}^+} f(t) t^{-iy}\,\frac{\mathrm{d}t}{t} \\
&= \int_{\mathbb{R}^+} f(t) t^{s}\,\frac{\mathrm{d}t}{t} \quad \text{letting } s = -iy.
\end{aligned}$$
If $f(t)$ grows at most as $t^{-\sigma_o}$ as $t \to 0^+$ and if $f(t)$ decreases rapidly as $t \to \infty$, then the integral converges on the complex open right half plane $\{\mathrm{Re}(s) > \sigma_o\}$. Thus we are led to define the *Mellin transform of* f for such functions f
$$\mathcal{M}f : \{\mathrm{Re}(s) > \sigma_o\} \longrightarrow \mathbb{C}, \quad (\mathcal{M}f)(s) = \int_{\mathbb{R}^+} f(t) t^{s}\,\frac{\mathrm{d}t}{t}.$$
The condition that $(\mathcal{F}\widetilde{f})(y)$ is small for large $|y|$ says that $(\mathcal{M}f)(s)$ is small for s far from the real axis.

For example, the gamma function is the Mellin transform of the negative exponential,

$$\Gamma(s) = \int_{\mathbb{R}^+} e^{-t} t^s \, \frac{\mathrm{d}t}{t}, \quad \mathrm{Re}(s) > 0.$$

Letting $g = \mathcal{M}f$ (so that $g(s) = \mathcal{M}f(s) = (\mathcal{F}\widetilde{f})(y)$ when $s = -iy$), the next question is how to recover f from g. Since g is simply the Fourier transform of f up to a coordinate change, f must be essentially the inverse Fourier transform of g. More specifically, the fact that $\widetilde{f}$ is exactly the inverse Fourier transform of $\mathcal{F}\widetilde{f}$,

$$\widetilde{f}(x) = \frac{1}{2\pi} \int_{\mathbb{R}} (\mathcal{F}\widetilde{f})(y) e^{iyx} \, \mathrm{d}y,$$

rewrites as

$$\begin{aligned} f(e^x) &= \frac{1}{2\pi} \int_{\mathbb{R}} g(s)(e^x)^{-s} \, \mathrm{d}y && \text{where } s = -iy \\ &= \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=0} g(s)(e^x)^{-s} \, \mathrm{d}s && \text{integrating upwards.} \end{aligned}$$

That is,

$$f(t) = \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=0} g(s) t^{-s} \, \mathrm{d}s.$$

Contour integration shows that the vertical line of integration can be shifted horizontally within the right half plane of convergence with no effect on the integral. Thus the definition of the *inverse Mellin transform* of g is inevitably

$$\mathcal{M}^{-1}g : \mathbb{R}^+ \longrightarrow \mathbb{C}, \quad (\mathcal{M}^{-1}g)(t) = \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=\sigma} g(s) t^{-s} \, \mathrm{d}s \quad \text{for any suitable } \sigma.$$

Naturally, the *Mellin inversion formula* says that if the functions f and g are well enough behaved then $g = \mathcal{M}f$ if and only if $f = \mathcal{M}^{-1}g$.

For practice with Mellin inversion, it is an exercise to evaluate the integral

$$f(t) = \int_{s=\sigma-i\infty}^{\sigma+i\infty} \Gamma(s) t^{-s} \, \mathrm{d}s, \quad \sigma > 0.$$