LOCAL FACTORS OF ZETA FUNCTIONS

We have seen that the functional equation of the Euler–Riemann zeta function is

$$\xi(1-s) = \xi(s), \quad s \in \mathbb{C},$$

where the *initial* definition of ξ is

$$\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s), \quad \mathrm{Re}(s) > 1.$$

The Euler product expression of $\zeta(s)$ is

$$\zeta(s) = \prod_{p \in \mathcal{P}} (1 - p^{-s})^{-1}.$$

And we have seen similar formulas in the context of Dirichlet L-functions.

This writeup sketches the calculation that the *Archimedean factors* such as $\pi^{-s/2}\Gamma(s/2)$ and the *non-Archimedean factors* such as $(1-p^{-s})^{-1}$ have a uniform description.

1. The Local Zeta Integral

Let $\mathbf{k}$ be any of the following fields:

$$\begin{cases} \mathbf{k} = \mathbb{Q}_p \text{ for some prime } p, \\ \mathbf{k} = \mathbb{R}, \\ \mathbf{k} = \mathbb{C}. \end{cases}$$

The **Schwartz space** of $\mathbf{k}$ is a collection of nice functions

$$\varphi : \mathbf{k} \longrightarrow \mathbb{C}.$$

Specifically, the Schwartz space consists of the smooth functions on $\mathbf{k}_v$ all of whose derivatives are rapidly decreasing. If $\mathbf{k} = \mathbb{Q}_p$ then such functions are compactly supported.

Take a Schwartz function

$$\varphi : \mathbf{k} \longrightarrow \mathbb{C},$$

and take a continuous multiplicative character

$$\chi : \mathbf{k}^\times \longrightarrow \mathbb{C}^\times.$$

Let s be a complex parameter. Then the **local zeta integral** is

$$Z(s) = Z(\varphi, \chi, s) = \int_{\mathbf{k}^\times} |\alpha|^s \chi(\alpha)\varphi(\alpha)\, d\alpha.$$

The integral converges for all s in some open right half plane.

2. Nonarchimedean Places

Let $\mathbf{k} = \mathbb{Q}_p$ for some prime p. The ring of integers of $\mathbf{k}$ is $\mathbb{Z}_p$. Consider a character

$$\chi : \mathbb{Q}_p^\times \longrightarrow \mathbb{C}^\times, \quad \chi(\alpha) = |\alpha|^{s_0}$$

(in the context of a primitive Dirichlet character, these local characters arise for the primes p away from the Dirichlet character's conductor), and consider a Schwartz function

$$\varphi : \mathbb{Q}_p \longrightarrow \mathbb{C}, \quad \varphi = 1_{\mathbb{Z}_p}.$$

This *locally constant* function is smooth, due to the non-Archimedean nature of $\mathbb{Q}_p$. The local zeta integral is

$$Z(s) = \int_{\mathbb{Q}_p^\times} |\alpha|^s \chi(\alpha)\varphi(\alpha)\, d\alpha$$

Normalize the multiplicative Haar measure so that $\mu(\mathbb{Z}_p^\times) = 1$. Evaluating the local integral is straightforward,

$$Z(s) = \int_{\mathbb{Q}_p^\times/\mathbb{Z}_p^\times} \int_{\mathbb{Z}_p^\times} |\alpha\eta|^{s+s_0} \varphi(\alpha\eta)\, d\eta\, d\alpha = \sum_{\ell=0}^{\infty} |p^\ell|^{(s+s_0)} = (1-\chi(p)p^{-s})^{-1}.$$

Thus the Euler factor of $\zeta(s)$ or of $L(s,\chi)$ at p is the p-adic zeta integral.

3. Real Archimedean Places

Let $\mathbf{k} = \mathbb{R}$. Consider the trivial character and the Gaussian Schwartz function

$$\begin{aligned} \chi &: \mathbb{R}^\times \longrightarrow \mathbb{C}^\times, & \chi(\alpha) &= 1, \\ \varphi &: \mathbb{R} \longrightarrow \mathbb{C}, & \varphi(\alpha) &= e^{-\pi\alpha^2}. \end{aligned}$$

The local integral is

$$Z(s) = \int_{\mathbb{R}^\times} |\alpha|^s \chi(\alpha)\varphi(\alpha)\, d\alpha = 2\int_0^\infty t^s e^{-\pi t^2} \frac{dt}{t} = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right).$$

On the other hand, consider the sign character and the simplest odd Schwartz function that incorporates the Gaussian,

$$\begin{aligned} \chi &: \mathbb{R}^\times \longrightarrow \mathbb{C}^\times, & \chi(\alpha) &= \operatorname{sgn}(\alpha), \\ \varphi &: \mathbb{R} \longrightarrow \mathbb{C}, & \varphi(\alpha) &= \alpha e^{-\pi\alpha^2}. \end{aligned}$$

Then the local integral is

$$Z(s) = \int_{\mathbb{R}^\times} |\alpha|^s \chi(\alpha)\varphi(\alpha)\, d\alpha = 2\int_0^\infty t^{s+1} e^{-\pi t^2} \frac{dt}{t} = \pi^{-\frac{s+1}{2}} \Gamma\left(\frac{s+1}{2}\right).$$

Thus the remaining factor in the functional equation of $\zeta(s)$ or of $L(s,\chi)$ is the Archimedian zeta integral.

4. Complex Archimedean Places

Let $\mathbf{k} = \mathbb{C}$. The unitary characters are

$$\chi : \mathbb{C}^\times \longrightarrow \mathbf{T}, \qquad \chi(\alpha) = \left(\frac{\alpha}{|\alpha|}\right)^m \text{ where } m \in \mathbb{Z}.$$

The Haar measure is

$$d\alpha = \frac{d^+\alpha}{|\alpha|^2} = \frac{r\,dr\,d\theta}{r^2}.$$

Note that the change of variable $\rho = r^2$ gives $2r\,dr/r^2 = d\rho/\rho$.

If $m = 0$ then let $\varphi(\alpha) = e^{-\pi|\alpha|^2}$. The local integral is

$$Z(s) = \int_{\mathbb{C}^\times} |\alpha|_{\mathbb{C}}^s \chi(\alpha)\varphi(\alpha)\,d\alpha = 2\pi \int_0^\infty r^{2s} e^{-\pi r^2}\,\frac{r\,dr}{r^2} = \pi \cdot \pi^{-s}\Gamma(s).$$

If $m > 0$ then let $\varphi(\alpha) = \bar{\alpha}^m e^{-\pi|\alpha|^2}$. The local integral is

$$\begin{aligned} Z(s) &= \int_{\mathbb{C}^\times} |\alpha|_{\mathbb{C}}^s \chi(\alpha)\varphi(\alpha)\,d\alpha = 2\pi \int_0^\infty r^{2s+m} e^{-\pi r^2}\,\frac{r\,dr}{r^2} \\ &= \pi \cdot \pi^{-s-\frac{m}{2}}\Gamma\left(s + \frac{m}{2}\right). \end{aligned}$$

If $m < 0$ then let $\varphi(\alpha) = \alpha^{-m} e^{-\pi|\alpha|^2}$. The local integral is

$$\begin{aligned} Z(s) &= \int_{\mathbb{C}^\times} |\alpha|_{\mathbb{C}}^s \chi(\alpha)\varphi(\alpha)\,d\alpha = 2\pi \int_0^\infty r^{2s-m} e^{-\pi r^2}\,\frac{r\,dr}{r^2} \\ &= \pi \cdot \pi^{-s+\frac{m}{2}}\Gamma\left(s - \frac{m}{2}\right). \end{aligned}$$

Thus in all cases,

$$Z(s) = \pi \cdot \pi^{-s-\frac{|m|}{2}}\Gamma\left(s + \frac{|m|}{2}\right).$$

This zeta integral is not something that we recognize, but it occurs naturally in the context of number fields larger than $\mathbb{Q}$.

A more classical normalization is to multiply the measure by $2/\pi$ and to replace $e^{-\pi|\alpha|^2}$ by $e^{-2\pi|\alpha|^2}$ in the Schwartz functions. Then in particular when $m = 0$, the local integral is $2 \cdot (2\pi)^{-s}\Gamma(s)$, and by Legendre's duplication formula

$$\Gamma(z)\Gamma(z + 1/2) = 2^{1-2z}\pi^{1/2}\Gamma(2z)$$

this is the product of the even and odd real archimedean factors from a moment ago. The classical normalization fits tidily with the factorization of the Dedekind zeta function of a CM-extension.

The point of this writeup is *not* that the zeta integral artifically repackages the various Euler factors in a uniform way, but rather that the specific choices of characters and Schwartz functions that reproduced the familiar Euler factors are in fact mere placeholders.