HECKE CHARACTERS CLASSICALLY AND IDÈLICALLY

Hecke's original definition of a Größencharakter, which we will call a Hecke character from now on, is set in the classical algebraic number theory environment. The definition is as it must be to establish the analytic continuation and functional equation for a general number field L-function

$$L(\chi, s) = \sum_{\mathfrak{a}} \chi(\mathfrak{a}) \mathrm{N}\mathfrak{a}^{-s} = \prod_{\mathfrak{p}} (1 - \chi(\mathfrak{p}) \mathrm{N}\mathfrak{p}^{-s})^{-1}$$

analogous to Dirichlet L-functions. But the classical generalization of a Dirichlet character to a Hecke character is complicated because it must take units and nonprincipal ideals into account, and it is difficult to motivate other than the fact that it is what works. By contrast, the definition of a Hecke character in the idèlic setting is simple and natural. This writeup explains the compatibility of the two definitions. Most of the ideas here were made clear to me by a talk that David Rohrlich gave at PCMI in 2009. Others were explained to me by Paul Garrett.

The following notation is in effect throughout:

- k denotes a number field.
- $\mathcal{O}$ denotes its ring of integers.
- $\mathbb{J}$ denotes the idèle group of k.
- v denotes a place of k, nonarchimedean or archimedean.

Contents

1. A Multiplicative Group Revisited

This initial section is a warmup whose terminology and result will fit into what follows.

By analogy to Dirichlet characters, we might think of groups of the form

$$(\mathcal{O}/\mathfrak{f})^\times, \quad \mathfrak{f} \text{ an ideal of } \mathcal{O}$$

as the natural domains of characters associated to the number field k. This idea is naïve, because $\mathcal{O}$ needn't have unique factorization, but as a starting point we define a group that is naturally isomorphic to the group in the previous display.

The group that we will define is arguably an improvement over $(\mathcal{O}/\mathfrak{f})^\times$. It will take the form of a quotient of multiplicative subgroups of $k^\times$,

$$k(\mathfrak{f})/k_\mathfrak{f} \quad \text{(notation to be explained soon)},$$

rather than being the unit group of a quotient ring of $\mathcal{O}$. Whereas in $(\mathcal{O}/\mathfrak{f})^\times$ the inverse of a coset $x+\mathfrak{f}$ is generally not the coset $x^{-1}+\mathfrak{f}$ of the inverse, because x^{-1} needn't lie in $\mathcal{O}$ at all, inverses in $k(\mathfrak{f})/k_\mathfrak{f}$ will be natural because $k(\mathfrak{f})$ is to be a multiplicative group.

The notion of two integral ideals of $\mathcal{O}$ being coprime generalizes easily to fractional ideals of k. For any fractional ideal,

$$\mathfrak{a} = \prod_\mathfrak{p} \mathfrak{p}^{e_\mathfrak{p}}, \quad \text{each } e_\mathfrak{p} \in \mathbb{Z},\ e_\mathfrak{p} = 0 \text{ for almost all } \mathfrak{p},$$

we say that the maximal ideal $\mathfrak{p}$ *appears in* $\mathfrak{a}$ if $e_\mathfrak{p} \neq 0$. If $\mathfrak{b}$ is a second fractional ideal then we say that $\mathfrak{a}$ and $\mathfrak{b}$ are coprime if no $\mathfrak{p}$ appears in both $\mathfrak{a}$ and $\mathfrak{b}$. The condition that $\mathfrak{a}$ and $\mathfrak{b}$ are coprime is written $(\mathfrak{a},\mathfrak{b}) = 1$.

Let $\mathfrak{f}$ be a nontrivial ideal of $\mathcal{O}$; that is, $\mathfrak{f}$ is neither the zero ideal nor $\mathcal{O}$. The elements of $k^\times$ that generate fractional ideals coprime to $\mathfrak{f}$ form a subgroup,

$$k(\mathfrak{f}) = \{\alpha \in k^\times : \big((\alpha),\mathfrak{f}\big) = 1\}.$$

Thus the condition $\alpha \in k(\mathfrak{f})$ is $\nu_\mathfrak{p}((\alpha)) = 0$ for all $\mathfrak{p}$ that appear in $\mathfrak{f}$. To define a suitable quotient of $k(\mathfrak{f})$, first note that the set

$$k(\mathfrak{f})\mathfrak{f} = \{\delta \in k : \nu_\mathfrak{p}((\delta)) \geq \nu_\mathfrak{p}(\mathfrak{f}) \text{ for all } \mathfrak{p} \text{ that appear in } \mathfrak{f}\}.$$

has the following properties:

- $k(\mathfrak{f})\mathfrak{f}$ contains 0 (even though $k(\mathfrak{f})$ does not, because $\mathfrak{f}$ does).
- $k(\mathfrak{f})\mathfrak{f}$ is preserved under multiplication by $k(\mathfrak{f})$ (because $k(\mathfrak{f})$ is a group); in particular, it is closed under negation.
- $k(\mathfrak{f})\mathfrak{f}$ is closed under addition. (To see so, take any $\delta,\delta' \in k(\mathfrak{f})\mathfrak{f}$, and fix any maximal ideal $\mathfrak{p}$ of $\mathcal{O}$ that appears in $\mathfrak{f}$. Then $\nu_\mathfrak{p}((\delta)) \geq \nu_\mathfrak{p}(\mathfrak{f})$ and $\nu_\mathfrak{p}((\delta')) \geq \nu_\mathfrak{p}(\mathfrak{f})$, so that $\nu_\mathfrak{p}((\delta+\delta')) \geq \nu_\mathfrak{p}(\mathfrak{f})$ as well. Thus $\delta+\delta' \in k(\mathfrak{f})\mathfrak{f}$.)

These three properties of $k(\mathfrak{f})\mathfrak{f}$ show that the following definition gives an equivalence relation.

Definition 1.1 (Multiplicative Congruence)**.** *For a pair of nonzero field elements $\alpha,\beta \in k(\mathfrak{f})$, the condition*

$$\alpha = \beta \bmod^\times \mathfrak{f}$$

means

$$\beta - \alpha \in k(\mathfrak{f})\mathfrak{f}.$$

The nomenclature *multiplicative* congruence will be explained soon. If $\alpha,\beta \in k(\mathfrak{f})$ are integral, the condition $\alpha = \beta \bmod^\times \mathfrak{f}$ is $\beta - \alpha \in k(\mathfrak{f})\mathfrak{f} \cap \mathcal{O} = \mathfrak{f}$, which is to say $\alpha = \beta \bmod \mathfrak{f}$; so multiplicative congruence subsumes ordinary congruence in $\mathcal{O}$ away from $\mathfrak{f}$.

Beyond the three equivalence relation properties, two more properties of multiplicative congruence follow from the fact that $k(\mathfrak{f})\mathfrak{f}$ is preserved under multiplication by $k(\mathfrak{f})$: For any $\alpha,\beta,\gamma,\delta \in k(\mathfrak{f})$,

$$\text{if } \alpha = \beta \bmod^\times \mathfrak{f} \text{ and } \gamma = \delta \bmod^\times \mathfrak{f} \text{ then } \alpha\gamma = \beta\delta \bmod^\times \mathfrak{f},$$

and as a special case, for any $\alpha, \beta, \gamma \in k(\mathfrak{f})$,

$$\text{if } \alpha = \beta \bmod^\times \mathfrak{f} \text{ then } \alpha\gamma = \beta\gamma \bmod^\times \mathfrak{f}.$$

With the properties of multiplicative congruence established, we can define the subgroup $k_\mathfrak{f}$ of $k(\mathfrak{f})$ that will give the desired quotient group $k(\mathfrak{f})/k_\mathfrak{f}$,

$$k_\mathfrak{f} = 1 + k(\mathfrak{f})\mathfrak{f} = \{\alpha \in k^\times : \alpha = 1 \bmod^\times \mathfrak{f}\} \subset k(\mathfrak{f}).$$

To see that $k_\mathfrak{f}$ is a subgroup note that if $\alpha, \beta = 1 \bmod^\times \mathfrak{f}$ then $\alpha = \beta \bmod^\times \mathfrak{f}$, and we may multiply through by β^{-1} to get $\alpha\beta^{-1} = 1 \bmod^\times \mathfrak{f}$.

Given α and β in $k(\mathfrak{f})$, the equivalences

$$\beta - \alpha \in k(\mathfrak{f})\mathfrak{f} \iff \beta \in \alpha + k(\mathfrak{f})\mathfrak{f} = \alpha + \alpha k(\mathfrak{f})\mathfrak{f} \iff \beta/\alpha \in 1 + k(\mathfrak{f})\mathfrak{f} = k_\mathfrak{f}$$

show that alternatively we could have defined the multiplicative congruence relation $\alpha = \beta \bmod^\times \mathfrak{f}$ to mean $\beta/\alpha \in k_\mathfrak{f}$, or equivalently,

$$\alpha k_\mathfrak{f} = \beta k_\mathfrak{f} \text{ in } k(\mathfrak{f}).$$

The analogy between this and the usual condition $\alpha + \mathfrak{f} = \beta + \mathfrak{f}$ in $\mathcal{O}$ for ordinary congruence modulo $\mathfrak{f}$ (or between the conditions $\beta/\alpha \in k_\mathfrak{f}$ in $k(\mathfrak{f})$ and $\beta - \alpha \in \mathfrak{f}$ in $\mathcal{O}$) explains why we view the relation $\alpha = \beta \bmod^\times \mathfrak{f}$ as multiplicative congruence. Indeed, this is more than an analogy; because $k(\mathfrak{f}) \cap \mathcal{O} = \{\alpha \in \mathcal{O} : ((\alpha), \mathfrak{f}) = 1\}$ and $k(\mathfrak{f})\mathfrak{f} \cap \mathcal{O} = \mathfrak{f}$, we have:

$$\text{For } \alpha, \beta \in \mathcal{O} \text{ coprime to } \mathfrak{f}, \quad \alpha + \mathfrak{f} = \beta + \mathfrak{f} \text{ in } \mathcal{O} \iff \alpha k_\mathfrak{f} = \beta k_\mathfrak{f} \text{ in } k(\mathfrak{f}).$$

This does most of the work to establish a group isomorphism that incorporates how multiplicative congruence subsumes ordinary congruence.

Proposition 1.2. *Let $\mathfrak{f}$ be a nontrivial integral ideal of the integer ring $\mathcal{O}$. There is a natural isomorphism*

$$(\mathcal{O}/\mathfrak{f})^\times \xrightarrow{\sim} k(\mathfrak{f})/k_\mathfrak{f}, \qquad \alpha + \mathfrak{f} \longmapsto \alpha k_\mathfrak{f}.$$

Proof. The work carried out just before the statement of the propostion shows that the map is a well defined monomorphism. To see that the map surjects, we need to show that any element $\alpha k_\mathfrak{f}$ of $k(\mathfrak{f})/k_\mathfrak{f}$ has an integral representative $\alpha\beta$ coprime to $\mathfrak{f}$. Let the negative part of the principal ideal (α) be

$$(\alpha)_{\text{neg}} = \prod_\mathfrak{p} \mathfrak{p}^{e_\mathfrak{p}}, \quad \text{each } e_\mathfrak{p} < 0.$$

This is supported away from $\mathfrak{f}$. By the Sun-Ze theorem there exists $\beta \in \mathcal{O}$ satisfying the conditions

$$\beta = 1 \bmod \mathfrak{f}, \qquad \beta = 0 \bmod (\alpha)_{\text{neg}}^{-1}.$$

So $\alpha\beta$ lies in $\mathcal{O}$ and is coprime to $\mathfrak{f}$, and also $\alpha\beta k_\mathfrak{f} = \alpha k_\mathfrak{f}$ because $\beta \in 1 + \mathfrak{f} \subset k_\mathfrak{f}$. □

We end this discussion with two remarks.

First, the previous proposition extends to the ideal $\mathfrak{f} = \mathcal{O}$ by defining $k_\mathfrak{f}$ as $(1 + k(\mathfrak{f})\mathfrak{f}) \cap k^\times$ in all cases. But this spurious clutter is not worthwhile because the case $\mathfrak{f} = \mathcal{O}$ is degenerate, giving $k(\mathfrak{f}) = k_\mathfrak{f} = k^\times$, so that the quotient $k(\mathfrak{f})/k_\mathfrak{f}$ is trivial. In classical terms, $(\mathcal{O}/\mathcal{O})^\times$ is indeed the trivial group rather than an empty construct if one allows the one-element ring ($0 + \mathcal{O}$ is invertible modulo $\mathcal{O}$ because $0 = 1 \bmod \mathcal{O}$), but the one-element quotient ring makes this case anomalous.

Second, the proposition also applies if we replace the number field k by one of its nonarchimedean completions k_v. In this case the isomorphism works out to (exercise)

$$(\mathcal{O}_v/\mathfrak{p}_v^{e_v})^\times \cong \mathcal{O}_v^\times/(1+\mathfrak{p}_v^{e_v}), \quad e>0.$$

2. Hecke Characters Classically

Again let $\mathfrak{f}$ be an integral ideal, i.e., an ideal of $\mathcal{O}$. Define

$$\begin{aligned} I(\mathfrak{f}) &= \{\text{fractional ideals of } k \text{ coprime to } \mathfrak{f}\}, \\ P(\mathfrak{f}) &= \{\text{principal fractional ideals } (\alpha) \text{ of } k \text{ coprime to } \mathfrak{f}\}, \\ P_{\mathfrak{f}} &= \{\text{principal fractional ideals } (\alpha) \text{ of } k \text{ where } \alpha = 1 \bmod^\times \mathfrak{f}\}. \end{aligned}$$

Thus we have a diagram in which the vertical segments are containments and the horizontal maps take elements α to their ideals (α),

$$\begin{tikzcd} & I(\mathfrak{f}) \arrow[d, dash] \\ k(\mathfrak{f}) \arrow[r] \arrow[d, dash] & P(\mathfrak{f}) \arrow[d, dash] \\ k_{\mathfrak{f}} \arrow[r] & P_{\mathfrak{f}} \end{tikzcd}$$

Also we have a map

$$k^\times \longrightarrow (\mathbb{R}^\times)^{r_1} \times (\mathbb{C}^\times)^{r_2}, \quad \alpha \longmapsto 1 \otimes \alpha,$$

where we identify $\mathbb{R} \otimes k$ (tensoring over $\mathbb{Q}$) with $\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$ in the usual way. [For example, $\mathbb{R} \otimes k = \mathbb{R} \otimes \mathbb{Q}[X]/\langle f(X)\rangle = \mathbb{R}[X]/\langle f(X)\rangle$, and $f(X)$ factors over $\mathbb{R}$ as a product of linear and quadratic terms.] We invoke the fact that

$$1 \otimes k_{\mathfrak{f}} \text{ is dense in } (\mathbb{R}^\times)^{r_1} \times (\mathbb{C}^\times)^{r_2}.$$

Definition 2.1 (Classical Hecke Character, first definition)**.** *Let $\mathfrak{f}$ be a (nonzero) ideal of $\mathcal{O}$, and let*

$$\chi_\infty : (\mathbb{R}^\times)^{r_1} \times (\mathbb{C}^\times)^{r_2} \longrightarrow \mathbb{C}^\times$$

be a continuous character. Then the character

$$\chi : I(\mathfrak{f}) \longrightarrow \mathbb{C}^\times$$

is a **Hecke character with conductor $\mathfrak{f}$ and infinity-type** *χ_∞ if χ_∞ determines χ on $P_{\mathfrak{f}}$ by the rule*

$$\chi\big((\alpha)\big) = \chi_\infty^{-1}(1 \otimes \alpha) \quad \text{for all } \alpha \in k_{\mathfrak{f}}.$$

That is, the following diagram must commute:

$$\begin{tikzcd} & P_{\mathfrak{f}} \arrow[dr, "\chi"] & \\ k_{\mathfrak{f}} \arrow[ur, "\alpha \mapsto (\alpha)"] \arrow[dr, "\alpha \mapsto 1\otimes\alpha"'] & & \mathbb{C}^\times \\ & (\mathbb{R}^\times)^{r_1} \times (\mathbb{C}^\times)^{r_2} \arrow[ur, "\chi_\infty^{-1}"'] & \end{tikzcd}$$

In the Hecke character setting, characters need not be **unitary**. That is, their outputs need not lie in the complex unit circle group $\mathbb{T}$. Some authors use the words *character* for the unitary case and *quasicharacter* for the general case, but we do not.

Naturally, a classical Hecke character is **primitive** if it is not induced from another classical Hecke character with conductor $\mathfrak{f}' \mid \mathfrak{f}$. Every classical Hecke character is induced from a unique primitive classical Hecke character. We will see that the issue of primitivity disappears in the idèlic environment.

Next we show that because a classical Hecke character χ has an associated infinity-type χ_∞ that determines χ on principal ideals $(\alpha) \in P_\mathfrak{f}$, i.e., $\alpha \in k_\mathfrak{f}$, also χ has an associated character of a finite group,

$$\varepsilon : (\mathcal{O}/\mathfrak{f})^\times \longrightarrow \mathbb{T},$$

such that χ_∞ and ε together determine χ on the larger collection of principal ideals $(\alpha) \in P(\mathfrak{f})$, i.e., $\alpha \in k(\mathfrak{f})$. The domain $(\mathcal{O}/\mathfrak{f})^\times$ of ε was mentioned at the beginning of this writeup as a naïve possibility for the domain of a Hecke character χ. Now we see that because χ incorporates χ_∞, one missing ingredient was the infinity-type, and because the domain of χ is all of $I(\mathfrak{f})$ rather than only $P(\mathfrak{f})$, the other missing ingredient was the possibility of nonprincipal ideals.

To prove the assertion in the previous paragraph, let $n = |k(\mathfrak{f})/k_\mathfrak{f}|$, a finite number because $k(\mathfrak{f})/k_\mathfrak{f}$ is isomorphic to $(\mathcal{O}/\mathfrak{f})^\times$. Then for any $\alpha \in k^\times$,

$$\begin{aligned}\alpha \in k(\mathfrak{f}) &\implies \alpha^n \in k_\mathfrak{f}\\ &\implies \chi((\alpha))^n = \chi((\alpha^n)) = \chi_\infty^{-1}(\alpha^n) = \chi_\infty^{-1}(\alpha)^n\\ &\implies \chi((\alpha)) = \varepsilon(\alpha)\chi_\infty^{-1}(\alpha) \quad \text{where } \varepsilon(\alpha)^n = 1,\end{aligned}$$

now letting $\chi_\infty^{-1}(\cdot)$ abbreviate $\chi_\infty^{-1}(1 \otimes \cdot)$. Because $\varepsilon(\alpha) = \chi((\alpha))\chi_\infty(\alpha)$ it follows that $\varepsilon : k(\mathfrak{f}) \longrightarrow \mathbb{T}$ is a character. Furthermore, ε is trivial on $k_\mathfrak{f}$ because χ is a classical Hecke character, so we may view ε as a character of $k(\mathfrak{f})/k_\mathfrak{f}$. Because $k(\mathfrak{f})/k_\mathfrak{f}$ is isomorphic to $(\mathcal{O}/\mathfrak{f})^\times$, this makes ε a character of $(\mathcal{O}/\mathfrak{f})^\times$. With this discussion in mind, we can rephrase the definition of a classical Hecke character:

Definition 2.2 (Classical Hecke Character, second definition)**.** *Let $\mathfrak{f}$ be a (nonzero) ideal of $\mathcal{O}$, and let*

$$\varepsilon : (\mathcal{O}/\mathfrak{f})^\times \longrightarrow \mathbb{T}$$

be a character, and let

$$\chi_\infty : (\mathbb{R}^\times)^{r_1} \times (\mathbb{C}^\times)^{r_2} \longrightarrow \mathbb{C}^\times$$

be a continuous character. Then the character

$$\chi : I(\mathfrak{f}) \longrightarrow \mathbb{C}^\times$$

is a **Hecke character with conductor $\mathfrak{f}$ and $(\mathcal{O}/\mathfrak{f})^\times$-type ε and infinity-type χ_∞** *if ε (viewed as a character of $k(\mathfrak{f})/k_\mathfrak{f}$) and χ_∞ determine χ on $P(\mathfrak{f})$ by the rule*

$$\chi\big((\alpha)\big) = \varepsilon(\alpha k_\mathfrak{f})\chi_\infty^{-1}(1 \otimes \alpha) \quad \textit{for all } \alpha \in k(\mathfrak{f}).$$

That is, the following diagram must commute:

$$\begin{tikzcd} & P(\mathfrak{f}) \arrow[dr, "\chi"] & \\ k(\mathfrak{f}) \arrow[ur, "\alpha\mapsto(\alpha)"] \arrow[dr, "\alpha\mapsto(\alpha k_{\mathfrak{f}},1\otimes\alpha)"'] & & \mathbb{C}^\times \\ & k(\mathfrak{f})/k_{\mathfrak{f}} \times (\mathbb{R}^\times)^{r_1} \times (\mathbb{C}^\times)^{r_2} \arrow[ur, "\varepsilon\cdot\chi_\infty^{-1}"'] & \end{tikzcd}$$

We have already argued that a classical Hecke character in the sense of Definition 2.1 is also a classical Hecke character in the sense of Definition 2.2. The converse holds as well because the diagram in Definition 2.2 restricts to the diagram in Definition 2.1.

3. Dirichlet Characters as Classical Hecke Characters

Especially, if $k=\mathbb{Q}$ and we are given a Dirichlet character with some period N,

$$\chi_{\mathrm{Dir}} : (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \mathbb{T},$$

then set $\mathfrak{f}=N\mathbb{Z}$ and note that $P(\mathfrak{f})$ is all of $I(\mathfrak{f})$. Note also that every fractional ideal of $\mathbb{Q}$ has a unique positive generator α, but the condition $(\alpha)\in P_{\mathfrak{f}}$ does not imply $\alpha\in\mathbb{Q}_{\mathfrak{f}}$ without the side condition $\alpha>0$. Define a Hecke character of ideals to be the Dirichlet character on positive generators,

$$\chi_{\mathrm{Hecke}} : P(\mathfrak{f}) \longrightarrow \mathbb{T}, \quad \chi_{\mathrm{Hecke}}\big((\alpha)\big) = \chi_{\mathrm{Dir}}(\alpha\,\mathrm{sgn}(\alpha)).$$

To verify that χ_{Hecke} is indeed a Hecke character with conductor $\mathfrak{f}$, we need to determine its $(\mathbb{Z}/N\mathbb{Z})^\times$-type and its infinity-type.

View χ_{Dir} as a character of $\mathbb{Q}(\mathfrak{f})/\mathbb{Q}_{\mathfrak{f}}$. For any $\alpha\in\mathbb{Q}_{\mathfrak{f}}$, going across the top of the diagram

$$\begin{tikzcd} & P(\mathfrak{f}) \arrow[dr, "\chi_{\mathrm{Hecke}}"] & \\ \mathbb{Q}(\mathfrak{f}) \arrow[ur] \arrow[rr, "\varepsilon\cdot\chi_\infty^{-1}"] & & \mathbb{C}^\times \end{tikzcd}$$

gives

$$\alpha \longmapsto \chi_{\mathrm{Hecke}}\left((\alpha)\right) = \chi_{\mathrm{Dir}}(\alpha\,\mathrm{sgn}(\alpha)) = \chi_{\mathrm{Dir}}(\alpha)\chi_{\mathrm{Dir}}(\mathrm{sgn}(\alpha)).$$

Thus the diagram commutes if we choose

$$\left\{\begin{aligned} \varepsilon(\alpha) &= \chi_{\mathrm{Dir}}(\alpha), \\ \chi_\infty^{-1}(\alpha) &= \chi_{\mathrm{Dir}}(\mathrm{sgn}(\alpha)) \end{aligned}\right\} \quad \alpha\in\mathbb{Q}(\mathfrak{f}).$$

Unsurprisingly, ε is simply the original Dirichlet character. To look more closely at χ_∞,

$$\chi_\infty^{-1}(\alpha) = \chi_{\mathrm{Dir}}(\mathrm{sgn}(\alpha)) = \begin{cases} 1 & \text{if } \chi_{\mathrm{Dir}} \text{ is even,} \\ \mathrm{sgn}(\alpha) & \text{if } \chi_{\mathrm{Dir}} \text{ is odd,} \end{cases} \quad \text{for } \alpha\in\mathbb{Q}_{\mathfrak{f}}.$$

That is, the infinity-type is the trivial character $\chi_\infty(x)=1$ for $x\in\mathbb{R}^\times$ if χ_{Dir} is even, and the infinity-type is the sign character $\chi_\infty(x)=\mathrm{sgn}(x)$ for $x\in\mathbb{R}^\times$ if χ_{Dir} is odd. The equality $\chi_{\mathrm{Dir}}(\alpha)=\chi((\alpha))\chi_\infty(\alpha)$ for $\alpha\in\mathbb{Q}(\mathfrak{f})$ shows how the Hecke ideal-character cannot see the sign of α but its infinity-type suitably reproduces any sign-sensitive behavior that the original Dirichlet character may have.

4. A Non-Dirichlet Classical Rational Hecke Character

Let I denote the multiplicative group of fractional ideals of $\mathbb{Q}$. For any complex number $s \in \mathbb{C}$, the character

$$\chi_s : I \longrightarrow \mathbb{C}^\times, \quad \chi_n((\alpha)) = |\alpha|^s$$

is well defined. In fact, χ_s is a Hecke character with trivial conductor $\mathfrak{f} = \mathbb{Z}$, with trivial ε-type (the only possible ε-type because $\mathbb{Q}(\mathbb{Z})/\mathbb{Q}_\mathbb{Z} = \mathbb{Q}^\times/\mathbb{Q}^\times$ is trivial), and with infinity-type

$$\chi_{\infty,s} : \mathbb{R}^\times \longrightarrow \mathbb{C}^\times, \quad \chi_{\infty,s}(\alpha) = |\alpha|^{-s}.$$

Indeed, we have $P(\mathfrak{f}) = I$ and $(r_1, r_2) = (1, 0)$, and so we need only to check that the following diagram commutes:

$$\begin{tikzcd}
& I \arrow[dr, "\chi_s"] & \\
\mathbb{Q}^\times \arrow[ur, "\alpha\mapsto(\alpha)"] \arrow[dr, "\alpha\mapsto 1\otimes\alpha"'] & & \mathbb{C}^\times \\
& \mathbb{R}^\times \arrow[ur, "\chi_{\infty,s}^{-1}"'] &
\end{tikzcd}$$

Because both paths across the diamond take α to $|\alpha|^s$, the diagram commutes as desired. Later in this writeup we will see a sense in which this Hecke character and many others like it are not particularly interesting.

5. A Family of Non-Dirichlet Classical Hecke Characters

Let $k = \mathbb{Q}(i)$, and let I denote the multiplicative group of fractional ideals of k. For any integer n, the character

$$\chi_n : I \longrightarrow \mathbb{C}^\times, \quad \chi_n((\alpha)) = (\alpha/|\alpha|)^{4n}$$

is well defined. In fact, χ_n is a Hecke character with trivial conductor $\mathfrak{f} = \mathcal{O}$, with trivial ε-type (the only possible ε-type because $k(\mathfrak{f})/k_\mathfrak{f} = k^\times/k^\times$ is trivial when $\mathfrak{f} = \mathcal{O}$), and with infinity-type

$$\chi_{\infty,n} : \mathbb{C}^\times \longrightarrow \mathbb{C}^\times, \quad \chi_{\infty,n}(\alpha) = (\alpha/|\alpha|)^{-4n}.$$

Indeed, we have $P(\mathfrak{f}) = I$ and $(r_1, r_2) = (0, 1)$, and so we need only to check that the following diagram commutes:

$$\begin{tikzcd}
& I \arrow[dr, "\chi_n"] & \\
k^\times \arrow[ur, "\alpha\mapsto(\alpha)"] \arrow[dr, "\alpha\mapsto 1\otimes\alpha"'] & & \mathbb{C}^\times \\
& \mathbb{C}^\times \arrow[ur, "\chi_{\infty,n}^{-1}"'] &
\end{tikzcd}$$

Because both paths across the diamond take α to $(\alpha/|\alpha|)^{4n}$, the diagram commutes as desired. Unlike the Hecke character in the previous section, the Hecke characters

χ_n are interesting: they help to establish a density result for Gaussian primes in a sector, the Gaussian integer counterpart to Dirichlet's theorem on rational primes in an arithmetic progression.

6. Hecke Characters Idèlically

The idèle topology is a colimit topology. For each finite set S of places of k that contains all the infinite places, form the topological product

$$\mathbb{J}_S = \prod_{v \in S} k_v^\times \times \prod_{v \notin S} \mathcal{O}_v^\times.$$

Then the definition of the idèles as a topological space is

$$\mathbb{J} = \mathrm{colim}_S \mathbb{J}_S.$$

(Alternatively, the adèle ring is a colimit as well,

$$\mathbb{A} = \mathrm{colim}_S \mathbb{A}_S, \qquad \mathbb{A}_S = \prod_{v \in S} k_v \times \prod_{v \notin S} \mathcal{O}_v,$$

and the adèlic unit group topology is inevitably the idèle topology. However, this is *not* the restriction of the adèle topology to the idèles.) By the nature of the idèle topology, the kernel of any continuous group homomorphism from $\mathbb{J}$ to $\mathbb{C}^\times$ contains almost all the local unit groups $\mathcal{O}_v^\times$.

The idèlic definition of a Hecke character is decisively simpler and more natural than the classical definition:

Definition 6.1 (Idèlic Hecke Character)**.** *A* **Hecke character** *of k is a continuous character of the idèle group of k that is trivial on $k^\times$,*

$$\chi : \mathbb{J} \longrightarrow \mathbb{C}^\times, \quad \chi(k^\times) = 1.$$

The continuity in the definition really should be understood without being mentioned, because we view $\mathbb{J}$ and $\mathbb{C}^\times$ as topological groups. From now on we freely omit reference to topology and continuity.

A Hecke character $\chi : \mathbb{J} \longrightarrow \mathbb{C}^\times$ has a conductor intrinsically built in, a product of local conductors at the finite places, even though its definition makes no direct reference to a conductor. We discuss this next.

At any nonarchimedean place v the local character $\chi_v : k_v^\times \longrightarrow \mathbb{C}^\times$ is determined by its values on the local units $\mathcal{O}_v^\times$ and by its value on a uniformizer ϖ_v. By the observation at the end of the first paragraph of this section, χ_v therefore takes the *unramified* form $\chi_v(x) = |x|_v^s$ (where $s \in \mathbb{C}$) for almost all nonarchimedean v.

If χ_v is unramified then the local conductor of χ is $\mathcal{O}_v$. If χ_v is ramified then the local conductor of χ is $\mathfrak{p}_v^{e_v}$ for the smallest $e_v > 0$ such that χ_v is defined on $\mathcal{O}_v^\times/(1+\mathfrak{p}_v^{e_v}) \cong (\mathcal{O}_v/\mathfrak{p}_v^{e_v})^\times$ (this isomorphism was discussed at the beginning of the writeup). The reason that any such e_v exists is that although there is a neighborhood of 1 in $\mathbb{C}^\times$ that contains no nontrivial subgroup, its inverse image under χ_v in the profinite unit group

$$\mathcal{O}_v^\times = \lim_{e_v} \mathcal{O}_v^\times/(1+\mathfrak{p}_v^{e_v}),$$

must contain a nontrivial subgroup $1+\mathfrak{p}_v^{e_v}$. Because the subgroup is mapped by χ_v to a subgroup, it lies in the kernel.

Fujisaki's Lemma states that the diagonal embedding of $k^\times$ in $\mathbb{J}$ is discrete, and the quotient of the *unit* idèles $\mathbb{J}^1$ by $k^\times$ is compact. Thus an idèlic Hecke character is a periodic function with discrete period. As such, it is amenable to Fourier analysis. In classical terms, the discreteness and compactness encode the structure theorem of the integer unit group $\mathcal{O}^\times$ and the finiteness of the class number of k.

Given an idèlic Hecke character, we show how to produce a corresponding classical Hecke character. Let the idèlic Hecke character be

$$\chi = \bigotimes_v \chi_v$$

and let its conductor be

$$\mathfrak{f} = \prod_v \mathfrak{p}_v^{e_v}.$$

Define a character of fractional ideals coprime to $\mathfrak{f}$,

$$\widetilde{\chi} : I(\mathfrak{f}) \longrightarrow \mathbb{C}^\times,$$

by the conditions

$$\widetilde{\chi}(\mathfrak{p}_v) = \chi_v(\mathcal{O}_v^\times \varpi_v), \quad \text{nonarchimedean } v \nmid \mathfrak{f}.$$

The conditions are sensible because the local characters are unramified away from the conductor. In order that $\widetilde{\chi}$ be a classical Hecke character, the composition

$$k_{\mathfrak{f}} \longrightarrow I(\mathfrak{f}) \xrightarrow{\widetilde{\chi}} \mathbb{C}^\times$$

needs to take the form $a \mapsto \widetilde{\chi}_\infty^{-1}(1 \otimes a)$ for some character $\widetilde{\chi}_\infty$ on $(\mathbb{R}^\times)^{r_1} \times (\mathbb{C}^\times)^{r_2}$. Compute that for any $a \in k_{\mathfrak{f}}$, with $(a) = \prod \mathfrak{p}_v^{\alpha_v}$, the composite is in fact (using the fact that χ is trivial on $k^\times$ at the last step)

$$a \longmapsto \prod \widetilde{\chi}(\mathfrak{p}_v)^{\alpha_v} = \prod \chi_v(\mathcal{O}_v^\times \varpi_v)^{\alpha_v} = \chi(a_{\text{fin}}) = \chi^{-1}(a_{\text{inf}}).$$

The natural identification of $\mathbb{J}_\infty$ and $(\mathbb{R}^\times)^{r_1} \times (\mathbb{C}^\times)^{r_2}$ takes a_{inf} to $1 \otimes a$. Thus, given an idèlic Hecke character χ, the corresponding character $\widetilde{\chi}$ of $I(\mathfrak{f})$ is a classical Hecke character whose infinite type matches that of the idèlic character,

$$\widetilde{\chi}_\infty = \chi_\infty.$$

The formula in the classical definition uses $\widetilde{\chi}_\infty^{-1}$ rather than $\widetilde{\chi}$ to produce this compatibility.

Conversely, given a classical Hecke character $\widetilde{\chi}$ of k having conductor $\mathfrak{f}$ and $(\mathcal{O}/\mathfrak{f})^\times$-type ε and infinity-type $\widetilde{\chi}_\infty$, we want a corresponding idèlic Hecke character χ.

- Because $1 \otimes k_{\mathfrak{f}}$ is dense in $\mathbb{R} \otimes k$, the infinite part χ_∞ of χ is determined by $\tilde{\chi}_\infty$.
- For $v \nmid \mathfrak{f}$, define χ_v by the condition
$$\chi_v(\mathcal{O}_v^\times \varpi_v) = \tilde{\chi}(\mathfrak{p}_v).$$
- Any $x \in \prod_{v|\mathfrak{f}} k_v^\times$ is closely approximated by some $\alpha \in k^\times$, and so the desired value $\chi(x)$ is closely approximated by $\prod_{v\nmid\mathfrak{f}} \chi_v^{-1}(\alpha_v)$ (including infinite v). Here we are using the requirement that $\chi = 1$ on $k^\times$.

If the classical Hecke character is imprimitive then the conductor of the resulting idèlic Hecke character is the conductor of the corresponding primitive classical Hecke character. Thus, as mentioned earlier, there is no such thing as an imprimitive idèlic Hecke character.

7. Dirichlet Characters as Idèlic Hecke Characters

In idèlic terms, a Hecke character of $\mathbb{Q}$ is a continuous character

$$\chi : \mathbb{J}_{\mathbb{Q}} \longrightarrow \mathbb{C}^\times, \quad \chi \text{ factors through } \mathbb{J}_{\mathbb{Q}}/\mathbb{Q}^\times.$$

The rational idèles decompose as

$$\mathbb{J}_{\mathbb{Q}} = \mathbb{Q}^\times \cdot \widehat{\mathbb{Z}}^\times \cdot \mathbb{R}_+^\times.$$

Indeed, given any rational idèle,

$$x = \big((u_p p^{e_p})_{p \text{ prime}}\,,\, r\big),$$

where each $u_p \in \mathbb{Z}_p^\times$, each $e_p \in \mathbb{Z}$, $e_p = 0$ for almost all p, and $r \in \mathbb{R}^\times$, there exists a unique nonzero rational number

$$\alpha = \pm \prod p^{-e_p} \in \mathbb{Q}^\times$$

such that

$$\alpha x = (u_p') \times r', \quad \text{ each } u_p' \in \mathbb{Z}_p^\times \text{ and } r \in \mathbb{R}_+^\times.$$

A Dirichlet character

$$\chi_{\text{Dir}} : (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \mathbb{T}$$

can also be viewed as a continuous character

$$\chi_{\text{Dir}} : \widehat{\mathbb{Z}}^\times \longrightarrow \mathbb{T}$$

because $(\mathbb{Z}/N\mathbb{Z})^\times$ is a quotient of $\widehat{\mathbb{Z}}^\times$, and because the topology of $\widehat{\mathbb{Z}} = \lim_N \mathbb{Z}/N\mathbb{Z}$ makes the pulled-back χ_{Dir} continuous. Thus the Dirichlet character gives rise to a Hecke character of the rational idèles,

$$\chi_{\text{Hecke}}(\alpha u t) = \chi_{\text{Dir}}(u), \quad \alpha \in \mathbb{Q}^\times,\ u \in \widehat{\mathbb{Z}}^\times,\ t \in \mathbb{R}_+^\times.$$

More specifically, if $N = \prod p^{e_p}$ then the Dirichlet character decomposes correspondingly via the Sun-Ze theorem as

$$\chi_{\text{Dir}} = \bigotimes \chi_{\text{Dir},p}, \quad \text{each } \chi_{\text{Dir},p} : (\mathbb{Z}/p^{e_p}\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times,$$

and because each $(\mathbb{Z}/p^{e_p}\mathbb{Z})^\times$ is a quotient of $\mathbb{Z}_p^\times$ we may view the character instead as

$$\chi_{\text{Dir}} = \bigotimes \chi_{\text{Dir},p}, \quad \text{each } \chi_{\text{Dir},p} : \mathbb{Z}_p^\times \longrightarrow \mathbb{C}^\times,$$

8. Discretely Parametrized Hecke Characters

The Hecke characters form the dual group of the quotient of the idèle group by the multiplicative group of the field,

$$\{\text{Hecke characters}\} = (\mathbb{J}/k^\times)^*,$$

a topological group under the compact-open topology. (For any compact $K \subset \mathbb{J}/k^\times$ and any open $V \subset \mathbb{C}^\times$, let

$$\mathcal{O}_{K,V} = \{\chi : \chi(K) \subset V\}.$$

The compact-open topology of $(\mathbb{J}/k^\times)^*$ is the topology generated by all such sets.)

Let $\mathbb{J}^1$ denote the group of norm-1 idèles. The short exact sequence

$$1 \longrightarrow \mathbb{J}^1/k^\times \longrightarrow \mathbb{J}/k^\times \xrightarrow{|\cdot|} \mathbb{R}^+ \longrightarrow 1,$$

has dual sequence

$$1 \longrightarrow (\mathbb{R}^+)^* \longrightarrow (\mathbb{J}/k^\times)^* \longrightarrow (\mathbb{J}^1/k^\times)^* \longrightarrow 1,$$

showing that $(\mathbb{J}/k^\times)^*$ is the product of a discrete group $(\mathbb{J}^1/k^\times)^*$ of unitary characters (the group is discrete and the characters unitary because $\mathbb{J}^1/k^\times$ is compact) and the group $(\mathbb{R}^+)^*$, isomorphic to $\mathbb{C}$ (because its elements are $x \mapsto x^s$ for $s \in \mathbb{C}$).

To describe the decomposition specifically, split the sequences in a fairly (but not completely) natural way. Let $r = r_1 + 2r_2$ where r_1 is the number of real archimedean places and r_2 the number of complex ones, and use r to map $\mathbb{R}^+$ to $\mathbb{J}$ by a suitably-normalized infinite diagonal embedding,

$$\iota : \mathbb{R}^+ \longrightarrow \mathbb{J}, \quad \iota(x) = (x_v^{1/r})_{v|\infty}.$$

Because $|\iota(x)| = x$, indeed ι (with its outputs viewed as cosets) splits the first sequence. Now, given an idèle α, the decomposition

$$\alpha = \alpha_1 \cdot \iota(|\alpha|), \quad \alpha_1 = \alpha/\iota(|\alpha|) \in \mathbb{J}^1$$

descends to cosets. Correspondingly there is a unique decomposition of any Hecke character, suppressing cosets from the formula,

$$\chi(\alpha) = \chi_1(\alpha_1)|\alpha|^s, \quad \chi_1 \in (\mathbb{J}^1/k^\times)^*, \ s \in \mathbb{C}.$$

The $\mathbb{C}$-parametrized part $\alpha \mapsto |\alpha|^s$ of the character is not particularly interesting, and so sometimes it is the discretely parametrized unitary characters

$$\chi_1 : \mathbb{J}^1/k^\times \longrightarrow \mathbb{C}^\times$$

that are referred to as Hecke characters.

If $k = \mathbb{Q}$ then the discretely parametrized unitary characters are simply the Dirichlet characters.

To argue this, we first show that any continuous character

$$\chi : \widehat{\mathbb{Z}}^\times \longrightarrow \mathbb{C}^\times$$

is a Dirichlet character. The point here, as discussed earlier in connection with the conductor, is that there is a neighborhood of 1 in $\mathbb{C}^\times$ that contains no nontrivial subgroup, but its inverse image is a neighborhood of 1 in $\widehat{\mathbb{Z}}^\times$, which necessarily contains a subgroup

$$K = \prod_{p \in S}(1 + p^{e_p}\mathbb{Z}_p)\prod_{p \notin S}\mathbb{Z}_p^\times, \quad S \text{ a finite set of primes.}$$

The subgroup must map to $1_{\mathbb{C}}$, and so χ factors through the corresponding quotient,

$$\widehat{\mathbb{Z}}^\times/K = \prod_{p \in S}\mathbb{Z}_p^\times/(1 + p^{e_p}) \approx \prod_{p \in S}(\mathbb{Z}_p/p^{e_p}\mathbb{Z}_p)^\times \approx \prod_{p \in S}(\mathbb{Z}/p^{e_p}\mathbb{Z})^\times.$$

That is, χ can be viewed as a character of $(\mathbb{Z}/N\mathbb{Z})^\times$ where $N = \prod_{p \in S} p^{e_p}$.

Now, any discretely parametrized unitary character of the rational idèles takes the form

$$\chi(\alpha u t) = \chi_1(\alpha u \cdot t/|\alpha u t|) = \chi_1(u \cdot 1/|u|), \quad \alpha \in \mathbb{Q}^\times, \ u \in \widehat{\mathbb{Z}}^\times, \ t \in \mathbb{R}_+^\times.$$

(The calculation can eliminate t because there is only one infinite place, i.e., it is particular to $k = \mathbb{Q}$.) That is, if we define

$$\chi_D : \widehat{\mathbb{Z}}^\times \longrightarrow \mathbb{C}^\times, \quad \chi_D(u) = \chi_1(u \cdot 1/|u|),$$

then any discretely parametrized unitary character takes the form

$$\chi(\alpha u t) = \chi_D(u), \quad \alpha \in \mathbb{Q}^\times,\ u \in \widehat{\mathbb{Z}}^\times,\ t \in \mathbb{R}_+^\times.$$

By the previous paragraph, χ_D is a Dirichlet character.

The $\mathbb{Z}$-indexed family of Hecke characters that we saw earlier,

$$\chi_n : \{\text{fractional ideals of } \mathbb{Q}(i)\} \longrightarrow \mathbb{C}^\times, \quad \chi_n((\alpha)) = (\alpha/|\alpha|)^{4n},$$

are the simplest non-Dirichlet unitary Hecke characters.