A DENSITY RESULT FOR $\mathbb{Q}(i)$

Contents

1. Review

A key step in Dirichlet's proof that there are infinitely many primes in any viable arithmetic progression $a + N\mathbb{Z}_{\geq 0}$ is tacitly a simple case of *Fourier anlaysis.*

Recall the relevant environment. The multiplicative group modulo N,

$$G = (\mathbb{Z}/N\mathbb{Z})^\times,$$

has as its dual group the Dirichlet characters modulo N,

$$\begin{aligned} G^* &= \{\text{homomorphisms} : G \longrightarrow \mathbb{C}^\times\} \\ &= \{\chi : G \longrightarrow \mathbb{C}^\times \mid \chi(g_1 g_2) = \chi(g_1)\chi(g_2) \text{ for all } g_1, g_2 \in G\}. \end{aligned}$$

The orthogonality relations for Dirichlet characters are that for each $\chi \in G^*$,

$$\sum_{g \in G} \chi(g) = \begin{cases} |G| & \text{if } \chi = 1, \\ 0 & \text{otherwise,} \end{cases}$$

and for each $g \in G$,

$$\sum_{\chi \in G^*} \chi(g) = \begin{cases} |G^*| & \text{if } g = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Although the second orthogonality relation is sufficient for the Fourier analytic step of Dirichlet's argument, that step will be rewritten here as a fuller version of Fourier analysis. In the context of Dirichlet's argument alone, the generalization is gratuitous, but the point is that it will scale up to larger situations.

For any function $f : G \longrightarrow \mathbb{C}$, the **Fourier transform** of f is a corresponding function on the dual group,

$$\widehat{f} : G^* \longrightarrow \mathbb{C}, \quad \widehat{f}(\chi) = \frac{1}{\phi(N)} \sum_{x \in G} f(x)\chi(x^{-1}),$$

and then the **Fourier series of** f is

$$s_f : G \longrightarrow \mathbb{C}, \quad s_f(x) = \sum_{\chi \in G^*} \widehat{f}(\chi)\chi(x).$$

The second orthogonality relation shows that the Fourier series reproduces the original function,

$$\begin{aligned} s_f(x) &= \frac{1}{\phi(N)} \sum_{\chi \in G^*} \sum_{y \in G} f(y)\chi(xy^{-1}) \\ &= \frac{1}{\phi(N)} \sum_{y \in G} f(y) \sum_{\chi \in G^*} \chi(xy^{-1}) \\ &= f(x). \end{aligned}$$

No qualifications on the function f, and no convergence issues of any sort, are involved here since the group G is finite.

Returning to the Dirichlet proof, specalize the function f to pick off $a \pmod N$,

$$f(x) = \begin{cases} 1 & \text{if } x = a, \\ 0 & \text{otherwise.} \end{cases}$$

Then for any $\chi \in G^*$, the χth Fourier coefficient of f is simply

$$\widehat{f}(\chi) = \chi(a^{-1})/\phi(N),$$

and the relation $s_f(x) = f(x)$ is inevitably just the second orthogonality relation,

$$\frac{1}{\phi(N)} \sum_{\chi \in G^*} \chi(xa^{-1}) = \begin{cases} 1 & \text{if } x = a, \\ 0 & \text{otherwise.} \end{cases}$$

The Dirichlet proof is concerned with the sum $\sum_{p=a(N)} p^{-s}$. The indicator function f lets us take the sum over all primes instead and then replace f by its Fourier series $s_f = (1/\phi(N)) \sum_\chi \chi(a^{-1})\chi$ to get

$$\sum_{p=a(N)} p^{-s} = \sum_{p \in \mathcal{P}} f(p)p^{-s} = \frac{1}{\phi(N)} \sum_{\chi} \chi(a^{-1}) \sum_{p \in \mathcal{P}} \chi(p)p^{-s}.$$

Meanwhile, for each Dirichlet L-function

$$L(s,\chi) = \sum_{n \in \mathbb{Z}^+} \chi(n)n^{-s} = \prod_{p \in \mathcal{P}} (1 - \chi(p)p^{-s})^{-1}, \quad \mathrm{Re}(s) > 1,$$

we have

$$\log L(s,\chi) = \sum_{\substack{p \in \mathcal{P} \\ \nu \in \mathbb{Z}^+}} \nu^{-1}\chi(p^\nu)p^{-\nu s} = \sum_{p \in \mathcal{P}} \chi(p)p^{-s} + \sum_{\substack{p \in \mathcal{P} \\ \nu \geq 2}} \nu^{-1}\chi(p^\nu)p^{-\nu s},$$

where the second term has absolute value at most 1 by the argument in Euler's proof. By the last display of the previous paragraph, the linear combination of these logarithms whose coefficients are the Fourier coeffients of f is the desired sum plus a small error term,

$$\frac{1}{\phi(N)} \sum_{\chi} \chi(a^{-1}) \log L(s,\chi) = \sum_{p=a(N)} p^{-s} + \varepsilon, \quad |\varepsilon| < 1.$$

These calculations reproduce events from the Dirichlet proof, but now the role of Fourier analysis in forming the appropriate linear combination of the functions $\log L(s,\chi)$ is clear.

2. A Fourier series

Now replace the finite group $G = (\mathbb{Z}/N\mathbb{Z})^\times$ with the compact group $G = \mathbb{R}/\mathbb{Z}$. We may view G as the interval $[0, 1]$ with its endpoints identified, the operation being addition modulo 1. The dual group of unitary characters is (letting $\mathbb{T}$ denote the unit circle in the complex plane)

$$\begin{aligned} G^* &= \{\text{continuous homomorphisms} : G \longrightarrow \mathbb{T}\} \\ &= \{\psi_n : n \in \mathbb{Z}\} \quad \text{where } \psi_n(x) = e^{2\pi i n x}. \end{aligned}$$

Analogously to the second orthgonality relation, we suspect that

$$\sum_n \psi_n = \delta,$$

where δ is the *Dirac delta distribution*,

$$\int_G f(x)\delta(x)\,dx = f(0) \quad \text{for measurable functions } f \text{ that are continuous at } 0.$$

For any measurable function $f : G \longrightarrow \mathbb{C}$, the Fourier transform of f is a corresponding function on the dual group,

$$\widehat{f} : G^* \longrightarrow \mathbb{C}, \quad \widehat{f}(\psi_n) = \int_G f(x)\psi_n(-x)\,dx,$$

and then the Fourier series of f is formally

$$s_f = \sum_{n\in\mathbb{Z}} \widehat{f}(\psi_n)\psi_n.$$

Heuristically, the Fourier series should reproduce the original function at its points of continuity,

$$\begin{aligned} s_f(x) &= \sum_{n\in\mathbb{Z}} \int_G f(y)\psi_n(-y)\,dy\,\psi_n(x) \\ &= \int_G f(y) \sum_{n\in\mathbb{Z}} \psi_n(x-y)\,dy \\ &= f(x) \quad \text{if } f \text{ is continuous at } x. \end{aligned}$$

But in fact the issues are trickier than this, and there exist continuous functions on $\mathbb{R}/\mathbb{Z}$ whose Fourier series drastically fail to reproduce them. However, the Fourier series does reproduce the function at points of differentiability.

Let $0 \le a < b \le 1$, and replace the function f from before that picked off one point of the finite group $G = (\mathbb{Z}/N\mathbb{Z})^\times$ with the function f that picks off the subinterval $[a, b]$ of $G = \mathbb{R}/\mathbb{Z}$,

$$f : [0, 1] \longrightarrow \mathbb{C}, \quad f(x) = \begin{cases} 1 & \text{if } x \in [a, b], \\ 0 & \text{otherwise.} \end{cases}$$

For any integer n, the nth Fourier coefficient of f is

$$\widehat{f}(n) = \int_a^b \psi_n(-x)\,dx = \begin{cases} b-a & \text{if } n = 0, \\ -\frac{1}{2\pi i n}\left(e^{-2\pi i n b} - e^{-2\pi i n a}\right) & \text{if } n \ne 0. \end{cases}$$

The squarewave f is well enough behaved that its Fourier series reproduces it except at its jump-points,

$$f(x) = \sum_{n\in\mathbb{Z}} \widehat{f}(n)\psi_n(x), \quad x \neq a, b.$$

3. A Hecke L-function

The integer ring $\mathbb{Z}[i]$ of the number field $\mathbb{Q}(i)$ forms a unique factorization domain with unit group $\mathbb{Z}[i]^\times = \{\pm 1, \pm i\}$. Consequently, for any integer n the multiplicative function

$$\chi_n : \mathbb{Z}[i] - \{0\} \longrightarrow \mathbb{C}, \quad \chi_n(z) = (z/|z|)^{4n}$$

is also sensible as a function of nonzero ideals,

$$\chi_n(\mathfrak{a}) = \chi_n(g) \quad \text{where } \mathfrak{a} = \langle g\rangle.$$

The corresponding L-function for $\chi = \chi_n$ is formally

$$L(s,\chi) = \sum_{\mathfrak{a}} \chi(\mathfrak{a})\mathrm{N}\mathfrak{a}^{-s} = \prod_{\mathfrak{p}} (1 - \chi(\mathfrak{p})\mathrm{N}\mathfrak{p}^{-s})^{-1},$$

and its logarithm is formally

$$\log L(s,\chi) = \sum_{\mathfrak{p}} \sum_{\nu\in\mathbb{Z}^+} \nu^{-1}\chi(\mathfrak{p}^\nu)\mathrm{N}\mathfrak{p}^{-\nu s}.$$

Basic estimates show that

$$\sum_{\substack{\mathfrak{p}\\ \nu\geq 2}} |\nu^{-1}\chi(\mathfrak{p}^\nu)\mathrm{N}\mathfrak{p}^{-\nu s}| \leq 2\sum_{\substack{p\in\mathcal{P}\\ \nu\geq 2}} \nu^{-1}p^{-\nu s},$$

and as in the Dirichlet argument, the right side is bounded above by 2. Thus in terms of behavior as $s \to 1^+$,

$$\log L(s,\chi) \sim \sum_{\mathfrak{p}} \chi(\mathfrak{p})\mathrm{N}p^{-s}.$$

(*We won't need the previous result once we know that nontrivial Hecke L-functions are finite but nonzero at $s = 1$, but:*) Especially, in the case of the trivial character χ_0,

$$\log \zeta_{\mathbb{Q}(i)}(s) \sim \sum_{\mathfrak{p}} \mathrm{N}p^{-s}.$$

4. The Dirichlet trick revisited

We may view $\mathbb{Q}(i)$ as a dense subfield of $\mathbb{C}$, and we may view the quotient of the group $\mathbb{C}^\times$ by the Gaussian integer unit group $\mathbb{Z}[i]^\times = \{\pm 1, \pm i\}$ as the punctured first quadrant with its boundary rays identified. The circle $\mathbb{R}/\mathbb{Z}$ maps naturally to the quotient $\mathbb{C}^\times/\mathbb{Z}[i]^\times$,

$$\gamma : x \longmapsto e^{2\pi i x/4} \cdot \mathbb{Z}[i]^\times.$$

Just as we view $\mathbb{R}/\mathbb{Z}$ as the interval $[0,1]$ with its endpoints identified, we may view $\gamma(\mathbb{R}/\mathbb{Z}) = \mathbb{C}^\times/\mathbb{R}^+\mathbb{Z}[i]^\times$ as the first quarter-circle with its endpoints identified as well. Note that the identity $e^{2\pi i n x} = (e^{2\pi i n x/4})^{4n}$ is

$$\psi_n = \chi_n \circ \gamma, \quad n \in \mathbb{Z}.$$

By analogy to Dirichlet's theorem, suppose that we are interested in the primes lying in a sector

$$S_{a,b} = \{a \cdot \pi/2 \leq \arg(z) \leq b \cdot \pi/2\}, \quad 0 \leq a < b \leq 1.$$

Recall the function $f : \mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{C}$ that picks off the interval $[a, b]$. The related function $g : \mathbb{C}^\times/\mathbb{R}^+\mathbb{Z}[i]^\times \longrightarrow \mathbb{C}$ defined by the condition

$$f = g \circ \gamma$$

is the function that picks off the corresponding arc of the quarter-circle. Because the Fourier series of f is

$$s_f = \sum_n \widehat{f}(n)\psi_n = \sum_n \widehat{f}(n)\chi_n \circ \gamma,$$

the function g is consequently

$$g = \sum_n \widehat{f}(n)\chi_n.$$

It follows that

$$\sum_{\pi \in S_{a,b}} \mathrm{N}\pi^{-s} \sim \sum_{\substack{\pi \in S_{0,1} \\ \nu \in \mathbb{Z}^+ \\ \pi^\nu \in S_{a,b}}} \nu^{-1}\mathrm{N}\pi^{-\nu s} = \sum_{\pi,\nu} g((\pi/|\pi|)^\nu)\nu^{-1}\mathrm{N}\pi^{-\nu s}.$$

That is, replacing g by its Fourier series,

$$\sum_{\pi \in S_{a,b}} \mathrm{N}\pi^{-s} \sim \sum_{n \in \mathbb{Z}} \widehat{f}(n) \sum_{\pi,\nu} \nu^{-1}\chi_n(\pi^\nu)\mathrm{N}\pi^{-\nu s} = \sum_{n \in \mathbb{Z}} \widehat{f}(n) \log L(s, \chi_n).$$

If the nontrivial L-functions are finite but nonzero at $s = 1$ then furthermore,

$$\sum_{n \in \mathbb{Z}} \widehat{f}(n) \log L(s, \chi_n) \sim (b - a) \log \zeta_{\mathbb{Q}(i)}(s) \sim (b - a) \sum_{\mathfrak{p}} \mathrm{N}p^{-s}.$$

In sum,

$$\lim_{s \to 1^+} \frac{\sum_{\pi \in S_{a,b}} \mathrm{N}\pi^{-s}}{\sum_{\mathfrak{p}} \mathrm{N}p^{-s}} = b - a.$$

That is,

> *In terms of Dirichlet density, the Gaussian primes are distributed evenly through sectors.*

Although half of the primes of $\mathbb{Z}[i]$ are rational primes $\pi = q \equiv 2 \pmod 3$, their Dirichlet density is 0. The point is that such primes are small in the sense of having large norms $N\pi = q^2$.