THE BERNOULLI NUMBERS, POWER SUMS, AND ZETA VALUES

The Bernoulli numbers arise naturally in the context of computing the *power sums*

$$\begin{aligned}
&1^0 + 2^0 + \cdots + n^0 = n, \\
&1^1 + 2^1 + \cdots + n^1 = \frac{1}{2}(n^2 + n), \\
&1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}(2n^3 + 3n^2 + n), \\
&\text{etc.}
\end{aligned}$$

Also they appear in Euler's evaluation of the zeta function $\zeta(k)$ at even integers $k \geq 2$, and then of the divergent series $\zeta(1-k)$ for the same k-values.

Contents

1. The Bernoulli numbers and power sums

Let n be a positive integer, and introduce notation for the kth power sum from 0 up to $n-1$ for any nonnegative integer k,

$$S_k(n) = \sum_{m=0}^{n-1} m^k, \quad k \in \mathbb{N}.$$

Thus $S_0(n) = n$ because $0^0 = 1$, while for $k \geq 1$ the term 0^k of $S_k(n)$ is 0. In particular, the second and third of the three summations shown above evaluate $S_1(n+1)$ and $S_2(n+1)$ but the first is not $S_0(n+1)$. (*Taking the power sum over m from 0 to $n-1$ as we are doing, rather than from 1 to n as is perhaps more natural, leads to the most common contemporary definition of the Bernoulli numbers, but the italicized parenthetical comments to follow will show that their definition is almost unchanged if instead the power sum does go from 1 to n.*) The power series having the power sums as its coefficients is their *generating function*, dependent on n and on a formal variable t, but gathering together the kth power sums for all k,

$$\mathbb{S}(n,t) = \sum_{k=0}^{\infty} S_k(n) \frac{t^k}{k!}\,.$$

Rearrange this double sum to get that $\mathbb{S}(n,t)$ equals $\sum_{m=0}^{n-1}\sum_{k=0}^{\infty}(mt)^k/k!$, which is the finite geometric sum $\sum_{m=0}^{n-1} e^{mt}$, having closed form $(e^{nt}-1)/(e^t-1)$, and then divide and multiply by t to get a product of two terms,

$$\mathbb{S}(n,t) = \frac{e^{nt}-1}{t}\,\frac{t}{e^t-1}.$$

(*If the power sum is taken instead over m from 1 to n then this is multiplied by e^t, and so the second term becomes $te^t/(e^t-1)$, which is $t/(1-e^{-t})$.*) The first term has the power series expansion

$$\frac{e^{nt}-1}{t} = \sum_{i=1}^{\infty}\frac{(nt)^i}{i!t} = \sum_{i=0}^{\infty}\frac{n^{i+1}}{i+1}\frac{t^i}{i!}\,.$$

The second term is independent of n. It is a power series in t whose coefficients are by definition the *Bernoulli numbers*, constants that can be computed once and for all as will be explained further below,

$$\boxed{\frac{t}{e^t-1} = \sum_{j=0}^{\infty} B_j\frac{t^j}{j!}\,.}$$

(*If the power sum is taken instead over m from 1 to n then $t/(1-e^{-t})$ rather than $t/(e^t-1)$ is defined as the power series with coefficients $B_j/j!$; because $t/(1-e^{-t})-t/(e^t-1)=t$, the only effect of this change on the Bernoulli numbers is to increase B_1 by 1.*) Again rearrange the generating function, this time by using the general formula

$$\sum_{i=0}^{\infty} a_i\frac{t^i}{i!}\sum_{j=0}^{\infty} b_j\frac{t^j}{j!} = \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}\binom{k}{j}a_{k-j}b_j\right)\frac{t^k}{k!}$$

or, equivalently, summing over diagonal segments, to get

$$\mathbb{S}(n,t) = \sum_{i=0}^{\infty}\frac{n^{i+1}}{i+1}\frac{t^i}{i!}\sum_{j=0}^{\infty}B_j\frac{t^j}{j!} = \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}\binom{k}{j}\frac{n^{k+1-j}}{k+1-j}B_j\right)\frac{t^k}{k!}\,.$$

This rewrites as

$$\begin{aligned}\mathbb{S}(n,t) &= \sum_{k=0}^{\infty}\left[\frac{1}{k+1}\sum_{j=0}^{k}\binom{k+1}{j}B_j n^{k+1-j}\right]\frac{t^k}{k!}\\ &= \sum_{k=0}^{\infty}\left[\frac{1}{k+1}\left(\sum_{j=0}^{k+1}\binom{k+1}{j}B_j n^{k+1-j} - B_{k+1}\right)\right]\frac{t^k}{k!}\,.\end{aligned}$$

Thus, if we define the ℓth *Bernoulli polynomial* as

$$B_\ell(X) = \sum_{j=0}^{\ell}\binom{\ell}{j}B_j X^{\ell-j},\quad \ell\ge 0,$$

which again can be computed once and for all, or if we define them all at once by a generating function,

$$\boxed{\frac{te^{Xt}}{e^t-1} = \sum_{\ell=0}^{\infty}B_\ell(X)\frac{t^\ell}{\ell!}\,,}$$

then matching the coefficients of the definition of $\mathbb{S}(n,t)$ as $\sum_{k=0}^{\infty} S_k(n)t^k/k!$ and of the last expansion of $\mathbb{S}(n,t)$, three displays back, shows that the kth power sum is a polynomial of degree $k+1$ in n,

$$\boxed{S_k(n) = \frac{1}{k+1}(B_{k+1}(n) - B_{k+1}), \quad k \geq 0.}$$

Because k is fixed and we imagine n to be large or indeterminate, this expression of $S_k(n)$ as a sum of k terms is a notable simplication of its original expression as a sum of n terms. The polynomial $S_k(X)$ has leading term $X^{k+1}/(k+1)$, second term B_1X^k, and lowest term B_kX, because

$$B_{k+1}(X) - B_{k+1} = X^{k+1} + \binom{k+1}{1}B_1X^k + \cdots + \binom{k+1}{k}B_kX.$$

(*If the power sum $S_k(n)$ is taken instead over m from 1 to n then the boxed formula is unchanged but $S_k(n)$ and $B_{k+1}(n)$ and B_{k+1} are redefined; when $k \geq 1$ both sides are bigger by n^k, the left side because it has gained the term n^k and lost 0^k, and the right side because B_1 is incremented; when $k = 0$ both versions of the boxed formula say that $S_0(n) = n$.*)

The first few Bernoulli numbers are $B_0 = 1$, $B_1 = -1/2$ (*or $B_1 = 1/2$ if $S_k(n)$ is taken from 1 to n*), $B_2 = 1/6$, and $B_3 = 0$, and so the first few Bernoulli polynomials are

$$\begin{aligned} B_0(X) &= 1, \\ B_1(X) &= X - \frac{1}{2}, \\ B_2(X) &= X^2 - X + \frac{1}{6}, \\ B_3(X) &= X^3 - \frac{3}{2}X^2 + \frac{1}{2}X. \end{aligned}$$

For example, the boxed formula gives

$$1^2 + 2^2 + \cdots + n^2 = S_2(n+1) = \frac{B_3(n+1) - B_3}{3},$$

and indeed the right side works out to $(2n^3 + 3n^2 + n)/6$, as at the beginning of this writeup. (*The reader can check that in the variant setup where the power sum is taken from 1 to n, we have $1^2 + 2^2 + \cdots + n^2 = S_2(n) = (B_3(n) - B_3)/3$, and because now $B_3(X) = X^3 + \frac{3}{2}X^2 + \frac{1}{2}$ this again gives $(2n^3 + 3n^2 + n)/6$.*)

2. Computing the Bernoulli numbers

Because the Bernoulli numbers are defined by the formal power series expansion

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!},$$

they are calculable in succession by matching coefficients in the power series identity (again from $\sum_j a_j t^j/j! \sum_k b_k t^k/k! = \sum_n (\sum_{j+k=n} \binom{n}{k} a_j b_k) t^n/n!$)

$$t = (e^t - 1)\sum_{k=0}^{\infty} B_k \frac{t^k}{k!} = \sum_{n=1}^{\infty} \left(\sum_{k=0}^{n-1} \binom{n}{k} B_k \right) \frac{t^n}{n!},$$

i.e., the nth parenthesized sum is 1 if $n = 1$ and 0 otherwise. That is, $B_0 = 1$ and then

$$\binom{n}{0}B_0 + \binom{n}{1}B_1 + \cdots + \binom{n}{n-1}B_{n-1} = 0, \quad n \geq 2.$$

Thus the Bernoulli numbers are rational. Further, because the expression

$$\frac{t}{e^t - 1} + \frac{t}{2} = \frac{t}{2} \cdot \frac{e^t + 1}{e^t - 1} = \frac{t}{2} \cdot \frac{e^{t/2} + e^{-t/2}}{e^{t/2} - e^{-t/2}}$$

is even, it follows that $B_1 = -1/2$ and $B_k = 0$ for all other odd k. For example,

$$\begin{aligned}
1 &= \binom{1}{0}B_0 & &\text{so } B_0 = 1 \text{ (again)}\\
0 &= \binom{2}{0}1 + \binom{2}{1}B_1 & &\text{so } B_1 = -1/2 \text{ (again)}\\
0 &= \binom{3}{0}1 - \binom{3}{1}\frac{1}{2} + \binom{3}{2}B_2 & &\text{so } B_2 = 1/6
\end{aligned}$$

and we know that $B_3 = 0$, and

$$0 = \binom{5}{0}1 - \binom{5}{1}\frac{1}{2} + \binom{5}{2}\frac{1}{6} + \binom{5}{4}B_4 \quad \text{so } B_4 = -1/30,$$

and similarly $B_6 = 1/42$ and so on.

3. Denominators of the Bernoulli numbers

We prove a result first shown by von Staudt and Clausen, independently, in 1840:

$$\boxed{B_k + \sum_{p:p-1|k} \frac{1}{p} \quad \text{is an integer} \quad \text{for } k = 0, 1, 2, 4, 6, 8, \ldots.}$$

For $k = 0$, the assertion is that $B_0 \in \mathbb{Z}$ (no positive multiple of any $p - 1$ is 0), and for $k = 1$ it is that $B_1 + 1/2 \in \mathbb{Z}$, and both of these are true by observation. So we may take $k \geq 2$, k even. As an example, $B_{12} = -691/2730$ and the primes p such that $p - 1 \mid 12$ are $2, 3, 5, 7, 13$, and one can confirm that $-691/2730 + 1/2 + 1/3 + 1/5 + 1/7 + 1/13 = 1$.

To prove the boxed statement, let $k \geq 2$ be even. As explained in section 1, the relation $S_k(X) = \frac{1}{k+1}(B_{k+1}(X) - B_{k+1})$ is

$$S_k(X) = \frac{X^{k+1}}{k+1} + B_1 X^k + \cdots + \frac{k}{2}B_{k-1}X^2 + B_k X.$$

For any prime p and positive integer e this gives

$$p^{-e}S_k(p^e) = \frac{p^{ke}}{k+1} + B_1 p^{(k-1)e} + \cdots + \frac{k}{2}B_{k-1}p^e + B_k,$$

from which

$$(1) \qquad B_k - p^{-e}S_k(p^e) \quad \text{is } p\text{-integral in } \mathbb{Q} \text{ for large } e.$$

With B_k and $p^{-e}S_k(p^e)$ related, we next relate $p^{-e}S_k(p^e)$ and $p^{-1}S_k(p)$, thus relating B_k and $p^{-1}S_k(p)$. For any integer $d \geq 2$, each of $0, 1, \ldots, p^d - 1$ uniquely takes the form $r + qp^{d-1}$ with $0 \le r < p^{d-1}$ and $0 \le q < p$, and so, using the binomial theorem for the first congruence to follow,

$$\begin{aligned}
S_k(p^d) &= \sum_{r=0}^{p^{d-1}-1} \sum_{q=0}^{p-1} (r + qp^{d-1})^k \\
&\equiv \sum_{r=0}^{p^{d-1}-1} \sum_{q=0}^{p-1} (r^k + kr^{k-1}qp^{d-1}) \pmod{p^d} \\
&= p \sum_{r=0}^{p^{d-1}-1} r^k + kp^{d-1} \sum_{r=0}^{p^{d-1}-1} r^{k-1} \sum_{q=0}^{p-1} q \\
&= pS_k(p^{d-1}) + \frac{k}{2}(p-1)p^d \sum_{r=0}^{p^{d-1}-1} r^{k-1} \quad (\text{with } \frac{k}{2} \in \mathbb{Z}) \\
&\equiv pS_k(p^{d-1}) \pmod{p^d}.
\end{aligned}$$

Consequently,

$$p^{-d} S_k(p^d) - p^{-(d-1)} S_k(p^{d-1}) \quad \text{is integral for each } d \ge 2.$$

Telescope this result for $d = 2, \ldots, e$ for any $e \ge 2$ to get

$$p^{-e} S_k(p^e) - p^{-1} S_k(p) \quad \text{is integral for each } e \ge 2.$$

Combine (1) and this to get that

$$B_k - p^{-1} S_k(p) \quad \text{is } p\text{-integral in } \mathbb{Q}. \tag{2}$$

Now, $S_k(p)$ modulo p is the geometric sum $\sum_{i=0}^{p-2} g^{ik}$ with g a generator, giving

$$S_k(p) \equiv \begin{Bmatrix} -1 & \text{if } p-1 \mid k \\ 0 & \text{if } p-1 \nmid k \end{Bmatrix} \pmod{p}.$$

Thus $p^{-1}(S_k(p)+1)$ is integral if $p-1 \mid k$ and $p^{-1}S_k(p)$ is integral if $p-1 \nmid k$. This combines with (2) to give

$$\begin{Bmatrix} B_k + p^{-1} \\ B_k \end{Bmatrix} \quad \text{is } p\text{-integral in } \mathbb{Q} \text{ if } \quad \begin{Bmatrix} p-1 \mid k \\ p-1 \nmid k \end{Bmatrix}.$$

Finally, if p and p' are distinct primes then p^{-1} is p'-integral in $\mathbb{Q}$, and so

$$B_k + \sum_{p:p-1|k} p^{-1} \quad \text{is } p\text{-integral in } \mathbb{Q} \text{ for all } p,$$

making it an integer, as claimed.

This argument is due to Witt, as presented early in the book *Local Fields* by Cassels. Note that all occurrences of *is integral* in the argument could be replaced the weaker *is p-integral in* $\mathbb{Q}$; that is, the argument is essentially p-adic until the p-adic results for all p are gathered at the last step.

4. The Bernoulli numbers and zeta values

Euler famously evaluated the infinite negative power sums

$$\zeta(k) = \sum_{n=1}^{\infty} n^{-k}, \quad k \geq 2 \text{ even},$$

with k understood to be an integer, and then used his functional equation for ζ to evaluate the divergent series $\zeta(1-k)$ for those same k, the latter zeta values simpler than the former. We skim the ideas here, necessarily invoking an expansion of the cotangent function, the functional equation for ζ, and the symmetry formula and Legendre duplication formula for the gamma function.

To compute $\zeta(k)$ for even $k \geq 2$, first note an identity that we have essentially seen already above,

$$\pi z \cot \pi z = \pi i z \frac{e^{\pi i z} + e^{-\pi i z}}{e^{\pi i z} - e^{-\pi i z}} = \pi i z + \frac{2\pi i z}{e^{2\pi i z} - 1}\,.$$

The right side fits into the definition of the Bernoulli numbers, including the lone nonzero odd Bernoulli number $B_1 = -1/2$, giving

$$\pi z \cot \pi z = \pi i z + \sum_{k \geq 0} \frac{(2\pi i)^k}{k!} B_k z^k = \sum_{\substack{k \geq 0 \\ \text{even}}} \frac{(2\pi i)^k}{k!} B_k z^k. \tag{3}$$

But also the cotangent has a second expansion,

$$\pi z \cot \pi z = 1 + z \sum_{n \geq 1} \left(\frac{1}{z-n} + \frac{1}{z+n} \right) = 1 - 2 \sum_{n \geq 1} \frac{z^2}{n^2} \cdot \frac{1}{1 - z^2/n^2}.$$

Although this expansion plausibly reproduces the cotangent, the fact that it does so is not trivial. Nonetheless, taking the expansion as granted, it is

$$\pi z \cot \pi z = 1 - 2 \sum_{n,e \geq 1} \frac{z^{2e}}{n^{2e}} = 1 - 2 \sum_{\substack{k \geq 2 \\ \text{even}}} \zeta(k) z^k. \tag{4}$$

Because the two power series expansions (3) and (4) of $\pi z \cot \pi z$ must match,

$$\boxed{\zeta(k) = -\frac{(2\pi i)^k}{2\,k!} B_k, \quad k \geq 2 \text{ even.}}$$

In particular $\zeta(2) = \pi^2/6$, $\zeta(4) = \pi^4/90$, and $\zeta(6) = \pi^6/945$.

Alternatively to the approach taken here, one can obtain the boxed formula by complex contour integration, and then the second expansion of the cotangent follows.

Now we go from $\zeta(k)$ to $\zeta(1-k)$ for $k \geq 2$ even. The computation that for $\mathrm{Re}(s) > 0$,

$$\begin{aligned}
\Gamma(s) &= \int_0^1 e^{-t} t^s \, \frac{\mathrm{d}t}{t} + \int_1^\infty e^{-t} t^s \, \frac{\mathrm{d}t}{t} \\
&= \sum_{n \geq 0} \frac{(-1)^n}{n!} \int_0^1 t^{s+n-1} \, \mathrm{d}t + \int_1^\infty e^{-t} t^s \, \frac{\mathrm{d}t}{t} \\
&= \sum_{n \geq 0} \frac{(-1)^n}{n!(s+n)} + \int_1^\infty e^{-t} t^s \, \frac{\mathrm{d}t}{t}
\end{aligned}$$

shows that $\Gamma(s)$ is the sum of two expressions, the first of which extends meromorphically from $\mathrm{Re}(s) > 0$ to $\mathbb{C}$ and the second of which extends analytically to $\mathbb{C}$. So overall, Γ extends meromorphically to $\mathbb{C}$ with a simple pole of residue $(-1)^n/n!$ at each nonpositive integer $-n \leq 0$. The functional equation for the completed zeta function, featuring the completed gamma function,

$$\pi^{-(1-s)/2} \Gamma(\tfrac{1-s}{2}) \zeta(1-s) = \pi^{-s/2} \Gamma(\tfrac{s}{2}) \zeta(s), \quad s \in \mathbb{C},$$

after being multiplied through by $\Gamma(\frac{s+1}{2})$ combines with the gamma function identities

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)} \qquad \text{(symmetry)}$$

and

$$\Gamma\left(\tfrac{s}{2}\right) \Gamma\left(\tfrac{s+1}{2}\right) = \pi^{\frac{1}{2}} 2^{1-s} \Gamma(s) \qquad \text{(Legendre duplication formula)}$$

to give (exercise)

$$\zeta(1-s) = 2(2\pi)^{-s} \Gamma(s) \cos(\tfrac{\pi s}{2}) \zeta(s).$$

Now for $s = k$ with $k \geq 2$ an even integer, so that $\Gamma(s) = (k-1)!$ and $\cos(\frac{\pi s}{2}) = (-1)^{k/2}$, substitute the boxed value of $\zeta(k)$ above to get

$$\zeta(1-k) = -2(2\pi)^{-k} (k-1)! (-1)^{k/2} \frac{(2\pi i)^k}{2\, k!} B_k,$$

and almost everything cancels,

$$\boxed{\zeta(1-k) = -\frac{B_k}{k}, \quad k \geq 2 \text{ even.}}$$

This is tidier than the value of $\zeta(k)$, with no power of π and no factorial. For example, $\zeta(-1) = -1/12$, $\zeta(-3) = 1/120$, $\zeta(-5) = -1/252$, etc. For elaborate computations with the zeta function and its variants that have similar functional equations, it is an indispensable gain of ease—and of likely-correct results—to move to the tidy divergent region of the functional equation, work there, and then take the answer back to the region of convergence.

5. The Bernoulli numbers and zeta values by contour integration

For $\mathrm{Re}(s) > 1$ compute that

$$\Gamma(s) n^{-s} = \int_{t=0}^\infty e^{-nt} (nt)^s \frac{\mathrm{d}(nt)}{nt} \cdot n^{-s} = \int_{t=0}^\infty e^{-nt} t^s \frac{\mathrm{d}t}{t}, \quad n = 1, 2, 3, \ldots,$$

and then passing a sum through an integral and using the formula $\sum_{n\geq 1} r^n = 1/(r^{-1}-1)$ for $|r|<1$ gives

$$\sum_{n\geq 1}\Gamma(s)n^{-s} = \int_{t=0}^{\infty}\sum_{n\geq 1} e^{-nt}t^s\,\frac{\mathrm{d}t}{t} = \int_{t=0}^{\infty}\frac{t^s}{e^t-1}\,\frac{\mathrm{d}t}{t}\,.$$

That is,

$$\Gamma(s)\zeta(s) = \int_{t=0}^{\infty}\frac{t^s}{e^t-1}\,\frac{\mathrm{d}t}{t}\,,\quad \mathrm{Re}(s)>1.$$

Here the condition $\mathrm{Re}(s)>1$ is required for the integral to converge at its left endpoint, but it converges for all complex s at its right end. Now let the *Hankel contour* or *keyhole contour* H_ε traverse the top side of the positive real axis from $+\infty$ in to some small ε, then a counterclockwise circle of radius ε about 0, then the bottom side of the positive real axis from ε back out to ∞, and consider the complex integral

$$\int_{H_\varepsilon}\frac{z^s}{e^z-1}\,\frac{\mathrm{d}z}{z}.$$

This integral is an entire function of the complex variable s, and it is independent of ε for $0<\varepsilon<2\pi$. Because z has argument 0 on the inward portion of H_ε and argument 2π on the outward portion, and $z^s=|z|^s e^{i\arg(z)s}$, while the integral over the circle is roughly $\varepsilon^{\mathrm{Re}(s)-1}$, the value of this integral is, letting ε go to 0,

$$\int_{H_\varepsilon}\frac{z^s}{e^z-1}\,\frac{\mathrm{d}z}{z} = (e^{2\pi is}-1)\int_{t=0}^{\infty}\frac{t^s}{e^t-1}\,\frac{\mathrm{d}t}{t} = (e^{2\pi is}-1)\Gamma(s)\zeta(s).$$

For $s=2,3,4,\dots$ this says only that the left integral is 0, but for other s with $\mathrm{Re}(s)>1$, it says that

$$\zeta(s) = \frac{1}{\Gamma(s)(e^{2\pi is}-1)}\int_{H_\varepsilon}\frac{z^s}{e^z-1}\,\frac{\mathrm{d}z}{z}\,,\quad \mathrm{Re}(s)>1,\ s\notin\mathbb{Z}.$$

In fact this formula gives values for $\zeta(s)$ for all complex s other than the positive integers; we have just given the second of the two continuation arguments for ζ in Riemann's 11-page paper. In particular this formula gives calculable values for $\zeta(s)$ where s is a nonpositive integer, as follows. Whereas for $\mathrm{Re}(s)>1$ the integrals over the two rays of H_ε differ by a multiplicative constant rather than canceling, and the integral over the circular part of H_ε goes to 0 in the limit, for nonpositive integer s the ray integrals will cancel while the integral over the circle will contribute a residue.

For $s\sim 1-k$ with $k\in\{1,2,3,\dots\}$ we have on the right side of the previous display, noting that $e^{2\pi is}=e^{2\pi i(s+k-1)}$ with $s+k-1$ small,

$$\Gamma(s)\sim\frac{(-1)^{k-1}}{(k-1)!(s+k-1)}\qquad\text{and}\qquad e^{2\pi is}-1\sim 2\pi i(s+k-1),$$

so that, making a limit tacit,

$$\frac{1}{\Gamma(1-k)(e^{2\pi i(1-k)}-1)} = \frac{(-1)^{k-1}(k-1)!}{2\pi i}.$$

Also, for $s=1-k$ the inward and outward portions of $\int_{H_\varepsilon}\frac{z^s}{e^z-1}\frac{\mathrm{d}z}{z}$ cancel because for $x>0$ the quantities $(xe^{i2\pi})^s=x^se^{i2\pi s}$ and x^s are equal. The integral over the counterclockwise circle around 0 gives $2\pi i\,\mathrm{Res}_0(f)$ where $f(z) = z^{-k}/(e^z - 1)$, and because

$$\frac{z^{-k}}{e^z - 1} = z^{-k-1}\frac{z}{e^z - 1} = z^{-k-1}\sum_{\ell\geq 0}\frac{B_\ell}{\ell!}z^\ell$$

the residue at 0 is $B_k/k!$. Assemble these observations to obtain a value for $\zeta(1-k)$,

$$\zeta(1-k) = \frac{(-1)^{k-1}(k-1)!}{2\pi i}2\pi i\frac{B_k}{k!}\,,$$

which is to say,

$$\boxed{\zeta(1-k) = (-1)^{k-1}\frac{B_k}{k}\,, \quad k \geq 1.}$$

For $k = 1$, this gives $\zeta(0) = -1/2$. For $k \geq 2$ even, this reproduces the formula $\zeta(1-k) = -B_k/k$ from the previous section, with no need to know $\zeta(k)$ first. For $k \geq 3$ odd, this gives $\zeta(-2) = \zeta(-4) = \cdots = 0$.

The quantity on the right side of the boxed equality is a bit more simply $-B_k/k$ for $k \geq 2$ but not for $k = 1$. It would be $-B_k/k$ for $k = 1$ as well if we were to use the alternate definition of the Bernoulli numbers from section 1, which gives $B_1 = 1/2$ rather than $B_1 = -1/2$. So here is an instance where the other definition is arguably more natural.

The methods of this section extend to give the values $L(\chi, 1-k)$ of any Dirichlet L-function at the nonnegative integers in terms of more general Bernoulli numbers $B_{\chi,k}$ associated to the Dirichlet character χ.

6. Zeta values without Bernoulli numbers

The values of $\zeta(k)$ for even $k \geq 2$ can be expressed finitely in closed form, with no reference to the Bernoulli numbers, if one is willing to admit an unwieldy formula with complex roots of unity instead. The formula has no real utility—we work it out only to show that it can be done—and so this section is entirely optional.

Consider an entire function of a complex variable z, understood to take the value 1 at $z = 0$, and well known to have both a product expansion and a Taylor series expansion that are valid for all $z \in \mathbb{C}$,

$$\varphi(z) = \frac{\sin(\pi z)}{\pi z} = \begin{cases}\prod_{n\geq 1}(1 - z^2/n^2)\\ \sum_{j\geq 0}(-1)^j\pi^{2j}z^{2j}/(2j+1)!.\end{cases}$$

The equality of the product and the series is

$$1 - \zeta(2)z^2 + \cdots = 1 - (\pi^2/6)z^2 + \cdots\,,$$

from which $\zeta(2) = \pi^2/6$.

Formulas for $\zeta(4), \zeta(6), \zeta(8), \ldots$ are obtained similarly. For any positive integer d, let $\rho_{2d} = e^{\pi i/d}$ be the first complex $2d$th root of unity and define

$$f_d(z) = \prod_{i=0}^{d-1}\varphi(\rho_{2d}^i z).$$

The product expansion of φ gives, rearranging a double product at the first equality,

$$f_d(z) = \prod_{n\geq 1}\prod_{i=0}^{d-1}\left(1 - \frac{\rho_d^i z^2}{n^2}\right) = \prod_{n\geq 1}\left(1 - \frac{z^{2d}}{n^{2d}}\right) = 1 - \zeta(2d)z^{2d} + \cdots\,.$$

And the Taylor series expansion of φ gives

$$f_d(z) = \prod_{i=0}^{d-1} \sum_{j_i=0}^{d} (-1)^{j_i} \frac{\rho_d^{ij_i}}{(2j_i+1)!} \pi^{2j_i} z^{2j_i} + \cdots .$$

Because the two expressions for $f_d(z)$ match, the sum-expansion of $f_d(z)$ must also begin with the constant term 1 and then zeros until its z^{2d} term,

$$f_d(z) = 1 + (-1)^d \sum_{j_0+j_1+\cdots+j_{d-1}=d} \frac{\rho_d^{j_1+2j_2+\cdots+(d-1)j_{d-1}}}{(2j_0+1)!(2j_1+1)!\cdots(2j_{d-1}+1)!} \pi^{2d} z^{2d} + \cdots .$$

Because the coefficients of the z^{2d} terms must match, $\zeta(2d)$ is an elementary sum of $\binom{2d-1}{d-1}$ terms,

$$\boxed{\zeta(2d) = (-1)^{d-1} \sum_{j_0+j_1+\cdots+j_{d-1}=d} \frac{\rho_d^{j_1+2j_2+\cdots+(d-1)j_{d-1}}}{(2j_0+1)!(2j_1+1)!\cdots(2j_{d-1}+1)!} \pi^{2d}}$$

And we may sum only the real parts of the summands. This formula is unaffected if ρ_d is replaced by a different primitive complex dth root of unity, and so it shows that $\zeta(2d)$ is a rational multiple of π^{2d}.

The boxed formula for $\zeta(2d)$ rapidly becomes intractable, asymptotically having $\mathcal{O}(4^d/\sqrt{d})$ terms involving ever-higher roots of unity. The formula $\zeta(2d) = (-1)^{d-1}\frac{2^{2d-1}B_{2d}}{(2d)!}\pi^{2d}$ is much more efficient. But still we can test the boxed formula for small values of d. As just above, for $d=1$ it says that

$$\zeta(2) = (-1)^0 \frac{1^1}{3!}\pi^2 = \frac{\pi^2}{6} .$$

For $d=2$ it says that

$$\begin{aligned} \zeta(4) &= (-1)^1 \left(\frac{(-1)^{2+2\cdot 0}}{5!} + \frac{(-1)^{1+2\cdot 1}}{3!^2} + \frac{(-1)^{0+2\cdot 2}}{5!} \right) \pi^4 \\ &= -\left(\frac{2}{5!} - \frac{1}{3!^2} \right) \pi^4 = \left(\frac{1}{36} - \frac{1}{60} \right) \pi^4 = \frac{\pi^4}{90} . \end{aligned}$$

For $d=3$ the vectors $j = (j_0, j_1, j_2)$ and the summands that they determine are (omitting π^6)

$$\begin{aligned} (3,0,0), (0,3,0), (0,0,3), (1,1,1) &: \frac{3}{7!} + \frac{1}{3!^3} \\ (2,1,0), (0,2,1), (1,0,2) &: \frac{3\rho_3}{5!3!} \\ (1,2,0), (2,0,1), (0,1,2) &: \frac{3\rho_3^2}{5!3!} . \end{aligned}$$

Because $\rho_3 + \rho_3^2 = -1$, the sum is

$$\zeta(6) = \left(\frac{3}{7!} + \frac{1}{3!^3} - \frac{3}{5!3!} \right) \pi^6 = \frac{\pi^6}{945} .$$

For $d = 4$ we freely may consider only the terms whose numerators $i^{j_1+2j_2+3j_3} = (-1)^{j_2} i^{j_1-j_3}$ are real, which is to say that j_1 and j_3 have the same parity. The relevant vectors (j_0, j_1, j_2, j_3) and the summands that they determine are (omitting π^8)

$$\begin{aligned}
(4,0,0,0),(0,4,0,0),(0,0,4,0),(0,0,0,4) &: \quad \frac{4}{9!} \\
(3,0,1,0),(0,3,0,1),(1,0,3,0),(0,1,0,3) &: -\frac{4}{7!3!} \\
(2,2,0,0),(2,0,2,0),(2,0,0,2),(0,2,2,0),(0,2,0,2),(0,0,2,2) &: -\frac{2}{5!^2} \\
(2,1,0,1),(1,2,1,0),(0,1,2,1),(1,0,1,2) &: \quad \frac{4}{5!3!^2} \\
(1,1,1,1) &: -\frac{1}{3!^4}.
\end{aligned}$$

Because $(-1)^{4-1} = -1$ out in front of the sum, we have altogether

$$\zeta(8) = \left(-\frac{4}{9!} + \frac{4}{7!3!} + \frac{2}{5!^2} - \frac{4}{5!3!^2} + \frac{1}{3!^4}\right)\pi^8 = \frac{\pi^8}{9450}\,.$$

The method here is related to Euler's first evaluation of $\zeta(2)$ through $\zeta(12)$ in 1735. The product expansion of the sine function was not yet well established then, and rather than introduce roots of unity and symmetrize, Euler used algebra to extract higher zeta values from matched coefficients and lower ones. As an example in the same spirit, granting the product expansion of the sine function, the equality of

$$\frac{\sin(\pi z)}{\pi z} = 1 - \frac{\pi^2 z^2}{6} + \frac{\pi^4 z^4}{120} + \cdots$$

and

$$\frac{\sin(\pi z)}{\pi z} = \prod_{n\geq 1}\left(1 - \frac{z^2}{n^2}\right) = 1 - z^2\sum_n \frac{1}{n^2} + z^4 \sum_{m<n}\frac{1}{m^2n^2} + \cdots$$

gives $\zeta(2) = \pi^2/6$, as already noted, and then $\sum_{m<n} 1/(m^2n^2) = \pi^4/120$. Now the relation

$$\frac{\pi^4}{36} = \zeta(2)^2 = \sum_{m,n}\frac{1}{m^2n^2} = \zeta(4) + 2\sum_{m<n}\frac{1}{m^2n^2} = \zeta(4) + \frac{\pi^4}{60}$$

gives

$$\zeta(4) = \pi^4\left(\frac{1}{36} - \frac{1}{60}\right) = \frac{\pi^4}{90}\,.$$

Once Euler did have the product expansion for the sine function, he also had the series expansion for the cotangent function by logarithmic differentiation, and therefore he had no need to work further with the product to obtain the values of $\zeta(2d)$. See section III.XVIII of André Weil's *Number Theory: An Approach Through History* for more on Euler and the zeta function.