THE BERNOULLI NUMBERS, POWER SUMS, AND ZETA VALUES

The Bernoulli numbers arise naturally in the context of computing the power sums

\[ 1^0 + 2^0 + \cdots + n^0 = n, \]
\[ 1^1 + 2^1 + \cdots + n^1 = \frac{1}{2}(n^2 + n), \]
\[ 1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}(2n^3 + 3n^2 + n), \]

etc.

Also they manifest in Euler’s evaluation of the zeta function \( \zeta(k) \) at even integers \( k \geq 2 \), and then of the divergent series \( \zeta(1-k) \) for the same \( k \)-values.

1. The Bernoulli Numbers and Power Sums

Let \( n \) be a positive integer, and introduce notation for the \( k \)th power sum up to \( n-1 \) for any nonnegative integer \( k \),

\[ S_k(n) = \sum_{m=0}^{n-1} m^k, \quad k \in \mathbb{N}. \]

Thus \( S_0(n) = n \) while for \( k \geq 1 \) the term \( 0^k \) of \( S_k(n) \) is 0. In particular, the second and third of the three summations shown above evaluate \( S_1(n+1) \) and \( S_2(n+1) \) but the first is not \( S_0(n+1) \). Having the sum start at 0 and stop at \( n-1 \) neats the ensuing calculation. The power series having these sums as its coefficients is their generating function,

\[ S(n,t) = \sum_{k=0}^{\infty} S_k(n) \frac{t^k}{k!}. \]

Rearrange the generating function by reversing the double sum and putting a finite geometric sum into closed form, and then divide and multiply by \( t \) to get

\[ S(n,t) = \frac{e^{nt} - 1}{t} \frac{t}{e^t - 1}. \]

The first term has the power series expansion

\[ \frac{e^{nt} - 1}{t} = \sum_{\ell=1}^{\infty} \frac{(nt)^\ell}{\ell! t} = \sum_{\ell=0}^{\infty} \frac{n^{\ell+1}}{\ell + 1} \frac{t^\ell}{\ell!}. \]

The second term is independent of \( n \). It is a power series in \( t \) whose coefficients are by definition the Bernoulli numbers, constants that can be computed once and for all as will be explained further below,

\[ \frac{t}{e^t - 1} = \sum_{j=0}^{\infty} B_j \frac{t^j}{j!}. \]
Again rearrange the generating function, this time by using the general formula
\[
\sum_{\ell=0}^{\infty} a_{\ell} t^{\ell} \sum_{j=0}^{\infty} b_j t^j / j! = \sum_{k=0}^{\infty} \left( \sum_{j=0}^{k} \binom{k}{j} a_{k-j} b_j \right) t^k / k!
\]
(or, equivalently, summing over diagonal segments),
\[
S(n, t) = \sum_{\ell=0}^{\infty} \binom{n+1}{\ell+1} \sum_{j=0}^{\infty} B_j t^j / j!
\]
\[
= \sum_{k=0}^{\infty} \left( \sum_{j=0}^{k} \binom{k}{j} \frac{n^{k+1-j}}{k+1-j} B_j \right) t^k / k!
\]
\[
= \sum_{k=0}^{\infty} \left( \frac{1}{k+1} \sum_{j=0}^{k} \binom{k+1}{j} B_j n^{k+1-j} \right) t^k / k!.
\]
Thus, if we define the kth Bernoulli polynomial as
\[
B_k(X) = \sum_{j=0}^{k} \binom{k}{j} B_j X^{k-j}, \quad k \geq 0,
\]
which again can be computed once and for all, then matching the coefficients of the definition of \( S(n, t) \) and of the expression that we just derived for it shows that the kth power sum is a polynomial of degree \( k+1 \) in \( n \),
\[
S_k(n) = \frac{1}{k+1} (B_{k+1}(n) - B_{k+1}), \quad k \geq 0.
\]
Because \( k \) is fixed and we imagine \( n \) to be large or indeterminate, this expression of \( S_k(n) \) as a sum of \( k+1 \) terms is a notable simplification of its original expression as a sum of \( n \) terms. The polynomial \( S_k(X) \) has leading term \( X^{k+1}/(k+1) \) and lowest term \( B_k(X) \), because
\[
B_{k+1}(X) = X^{k+1} + \binom{k+1}{1} B_1 X^k + \cdots + \binom{k+1}{k} B_k X + B_{k+1}.
\]
The first few Bernoulli numbers are \( B_0 = 1, B_1 = -1/2, B_2 = 1/6, \) and \( B_3 = 0, \) and so the first few Bernoulli polynomials are
\[
B_0(X) = 1,
\]
\[
B_1(X) = X - \frac{1}{2},
\]
\[
B_2(X) = X^2 - X + \frac{1}{6},
\]
\[
B_3(X) = X^3 - \frac{3}{2} X^2 + \frac{1}{2} X.
\]
For example, the boxed formula gives
\[
1^2 + 2^2 + \cdots + n^2 = S_2(n+1) = \frac{B_3(n+1) - B_3}{3},
\]
and indeed the right side works out to \( (2n^3 + 3n^2 + n)/6 \), as at the beginning of this writeup.
2. Computing the Bernoulli Numbers

Because the Bernoulli numbers are defined by the formal power series expansion
\[
\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!},
\]
they are calculable in succession by matching coefficients in the power series identity
\[
t = (e^t - 1) \sum_{k=0}^{\infty} B_k \frac{t^k}{k!} = \sum_{n=1}^{\infty} \left( \sum_{k=0}^{n-1} \binom{n}{k} B_k \right) \frac{t^n}{n!},
\]
i.e., the nth parenthesized sum is 1 if \( n = 1 \) and 0 otherwise. Thus the Bernoulli numbers are rational. Further, because the expression
\[
t = (e^t - 1) \sum_{k=0}^{\infty} B_k \frac{t^k}{k!} = \sum_{n=1}^{\infty} \left( \sum_{k=0}^{n-1} \binom{n}{k} B_k \right) \frac{t^n}{n!},
\]
is even, it follows that \( B_1 = -1/2 \) and \( B_k = 0 \) for all other odd \( k \). For example,
\[
1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} B_0, \quad \text{so } B_0 = 1,
\]
\[
0 = \begin{pmatrix} 2 \\ 0 \end{pmatrix} B_0 + \begin{pmatrix} 2 \\ 1 \end{pmatrix} B_1, \quad \text{so } B_1 = -1/2 \text{ (again)},
\]
\[
0 = \begin{pmatrix} 3 \\ 0 \end{pmatrix} B_0 + \begin{pmatrix} 3 \\ 1 \end{pmatrix} B_1 + \begin{pmatrix} 3 \\ 2 \end{pmatrix} B_2, \quad \text{so } B_2 = 1/6,
\]
and similarly \( B_4 = -1/30, B_6 = 1/42, \) and so on.

In hindsight, there are advantages to a slightly variant definition of the Bernoulli numbers,
\[
\frac{te^t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}.
\]
Because \( \frac{te^t}{e^t - 1} - \frac{t}{e^t - 1} = t \), this produces the same Bernoulli numbers other than changing \( B_1 \) from \(-1/2\) to \(1/2\), and the latter value works better in some contexts.

3. Denominators of the Bernoulli Numbers

We prove a result first shown by von Staudt and Clausen, independently, in 1840:

\[
B_k + \sum_{p \mid p-1 \mid k} \frac{1}{p} \quad \text{is an integer} \quad \text{for } k = 0, 1, 2, 4, 6, 8, \ldots.
\]

For \( k = 0 \), the assertion is that \( B_0 \in \mathbb{Z} \) (no positive multiple of any \( p-1 \) is 0), and for \( k = 1 \) it is that \( B_1 + 1/2 \in \mathbb{Z} \), and both of these are true by observation. So we may take \( k \geq 2 \), \( k \) even. As an example, \( B_{12} = -691/2730 \) and the primes \( p \) such that \( p-1 \mid 12 \) are 2, 3, 5, 7, 13, and one can confirm that \(-691/2730 + 1/2 + 1/3 + 1/5 + 1/7 + 1/13 = 1\).

To prove the boxed statement, let \( k \geq 2 \) be even. For any integer \( d \geq 2 \), each of \( 0, 1, \ldots, p^d - 1 \) uniquely takes the form \( r + qp^{d-1} \) with \( 0 \leq r < p^{d-1} \) and \( 0 \leq q < p \).
and so, using the binomial theorem for the first congruence to follow,

\[ S_k(p^d) = \sum_{r=0}^{p^d-1} \sum_{q=0}^{p-1} (r + qp^{d-1})^k \]

\[ \equiv \sum_{r=0}^{p^d-1} \sum_{q=0}^{p-1} (r^{k} + k r^{k-1} q p^{d-1}) \pmod{p^d} \]

\[ = p \sum_{r=0}^{p^d-1} r^k \sum_{q=0}^{p-1} r^{k-1} \sum_{q=0}^{p-1} q \]

\[ = pS_k(p^{d-1}) + \frac{k}{2} (p - 1) p^d \sum_{r=0}^{p^d-1} r^{k-1} \]

\[ \equiv pS_k(p^{d-1}) \pmod{p^d}. \]

Consequently,

\[ p^{-d} S_k(p^d) - p^{-(d-1)} S_k(p^{d-1}) \]

is integral for each \( d \geq 2 \).

Telescope this result for \( d = 2, \ldots, e \) for any \( e \geq 2 \) to get

(1) \[ p^{-e} S_k(p^e) - p^{-1} S_k(p) \]

is integral for each \( e \geq 2 \).

Further, \( S_k(p) \) modulo \( p \) is the geometric sum \( \sum_{i=0}^{p-2} g^i \) with \( g \) a generator, giving

\[ S_k(p) \equiv \begin{cases} -1 & \text{if } p - 1 \mid k \\ 0 & \text{if } p - 1 \nmid k \end{cases} \pmod{p}. \]

Thus \( p^{-1} (S_k(p) + 1) \) is integral if \( p - 1 \mid k \) and \( p^{-1} S_k(p) \) is integral if \( p - 1 \nmid k \). This combines with (1) to give

(2) \[ \left\{ \begin{array}{l} p^{-e} S_k(p^e) + p^{-1} \\ p^{-e} S_k(p^e) \end{array} \right\} \]

is integral if \( \left\{ \begin{array}{l} p - 1 \mid k \\ p - 1 \nmid k \end{array} \right\} \).

As explained in section 1, \( S_k(X) = \frac{1}{k+1} (B_{k+1}(X) - B_k) \) is

\[ S_k(X) = \frac{X^{k+1}}{k+1} + B_1 X^k + \cdots + \frac{k}{2} B_{k-1} X^2 + B_k X, \]

so that

\[ p^{-e} S_k(p^e) = \frac{p^{ke}}{k+1} + B_1 p^{(k-1)e} + \cdots + \frac{k}{2} B_{k-1} p^e + B_k, \]

from which

\[ B_k - p^{-e} S_k(p^e) \]

is \( p \)-integral in \( \mathbb{Q} \) for large \( e \).

This combines with (2) for large \( e \) to give

\[ \left\{ \begin{array}{l} B_k + p^{-1} \\ B_k \end{array} \right\} \]

is \( p \)-integral in \( \mathbb{Q} \) if \( \left\{ \begin{array}{l} p - 1 \mid k \\ p - 1 \nmid k \end{array} \right\} \).

Finally, if \( p \) and \( p' \) are distinct primes then \( p^{-1} \) is \( p' \)-integral in \( \mathbb{Q} \), and so

\[ B_k + \sum_{p: p^{-1} \nmid k} p^{-1} \]

is \( p \)-integral in \( \mathbb{Q} \) for all \( p \),

making it an integer, as claimed.
This argument is due to Witt, as presented early in the book *Local Fields* by Cassels. Note that all occurrences of *is integral* in the argument could be replaced with the weaker *is p-integral in* \(\mathbb{Q}\); that is, the argument is essentially p-adic until the p-adic results for all \(p\) are gathered at the end.

4. The Bernoulli Numbers and Zeta Values

Euler famously evaluated the infinite negative power sums

\[ \zeta(k) = \sum_{n=1}^{\infty} n^{-k}, \quad k \geq 2 \text{ even}, \]

with \(k\) understood to be an integer, and then used his functional equation for \(\zeta\) to evaluate the divergent series \(\zeta(1 - k)\) for those same \(k\), the latter zeta values simpler than the former. We skim the ideas here, necessarily invoking an expansion of the cotangent function, the functional equation for \(\zeta\), the Legendre duplication formula for the gamma function, and the behavior of the gamma function at negative integers.

To compute \(\zeta(k)\) for even \(k \geq 2\), first note an identity that we have essentially seen already above,

\[ \pi z \cot \pi z = \pi iz + \sum_{k \geq 0} \frac{(2\pi i)^k}{k!} B_k z^k. \]

The right side fits into the definition of the Bernoulli numbers, including the lone nonzero odd Bernoulli number \(B_1 = -1/2\), giving

\[ \zeta(k) = -\left(\frac{2\pi i}{k!}\right)^k B_k, \quad k \geq 2 \text{ even}. \]

In particular \(\zeta(2) = \pi^2/6\), \(\zeta(4) = \pi^4/90\), and \(\zeta(6) = \pi^6/945\).

Alternatively to the approach taken here, one can obtain the boxed formula by complex contour integration, and then the second expansion of the cotangent follows.

Now we go from \(\zeta(k)\) to \(\zeta(1 - k)\) for \(k \geq 2\) even. The Legendre duplication formula for the gamma function is

\[ \Gamma(s)\Gamma(s + \frac{1}{2}) = \pi^{\frac{1}{2}} 2^{1-2s} \Gamma(2s). \]
Also, the computation that for $\text{Re}(s) > 0$,

$$
\Gamma(s) = \int_0^1 e^{-t^s} \frac{dt}{t} + \int_1^\infty e^{-t^s} \frac{dt}{t} = \sum_{n \geq 0} \frac{(-1)^n}{n!} \int_0^1 t^{s+n-1} dt + \int_1^\infty e^{-t^s} \frac{dt}{t} = \sum_{n \geq 0} \frac{(-1)^n}{n! t^{s+n}} + \int_1^\infty e^{-t^s} \frac{dt}{t}
$$

expresses $\Gamma(s)$ as the sum of two expressions, the first of which extends meromorphically from $\text{Re}(s) > 0$ to $\mathbb{C}$ and the second of which extends analytically to $\mathbb{C}$.

At any nonpositive integer $-n \leq 0$ the extended gamma function has a simple pole of residue $\left(-\frac{1}{2}\right)^n n!$, giving for $k \geq 2$ even,

$$
\Gamma(s) \sim \frac{1}{k!(s+k)} \quad \text{and} \quad \Gamma(s/2) \sim \frac{2(-1)^{k/2}}{(k/2)!(s+k)} \quad \text{as } s \to -k,
$$

and so we encode a limit as a finite quotient of two infinite values,

$$
\frac{\Gamma(-k/2)}{\Gamma(-k)} = \frac{2(-1)^{k/2} k!}{(k/2)!}, \quad k \geq 2 \text{ even}.
$$

The work to follow will “multiply and divide by infinity,” but doing so is only convenient abbreviation of a tacit and legitimate limit process for the sake of clarity.

The functional equation for the completed zeta function specializes to

$$
\pi^{-1/2} \Gamma\left(\frac{1-k}{2}\right) \zeta(1-k) = \pi^{-1/2} \Gamma\left(\frac{k}{2}\right) \zeta(k), \quad k \geq 2 \text{ even.}
$$

Multiply the specialized functional equation by $\Gamma(-k/2)$ and then divide it by $\pi^{1/2} \Gamma(-k/2)\Gamma((-1-k)/2)$, which is $\pi^{k+1/2} 2^{1+k} \Gamma(-k)$ by the Legendre duplication formula, to get for any even integer $k \geq 2$,

$$
\zeta(1-k) = \frac{\Gamma(-k/2)\Gamma(k/2)}{\pi^k 2^{1+k} \Gamma(-k)} \zeta(k).
$$

By our finite quotient of two infinite values, and because $\Gamma(k/2) = (k/2 - 1)!$, this is

$$
\zeta(1-k) = \frac{2(-1)^{k/2} k!(k/2 - 1)!}{\pi^k 2^{1+k} (k/2)!} \zeta(k) = \frac{(-1)^{k/2} k!}{\pi^k 2^k (k/2)!} \zeta(k).
$$

Substitute the boxed value of $\zeta(k)$ above to get

$$
\zeta(1-k) = -\frac{(-1)^{k/2} k!(2\pi i)^k}{\pi^k 2^k k!} B_k,
$$

and almost everything cancels,

$$
\zeta(1-k) = -\frac{B_k}{k}, \quad k \geq 2 \text{ even.}
$$

This is tidier than the value of $\zeta(k)$, with no power of $\pi$ and no factorial. For example, $\zeta(-1) = -1/12$, $\zeta(-3) = 1/120$, $\zeta(-5) = -1/252$, etc. For elaborate computations with the zeta function and its variants that have similar functional equations, it is an indispensable gain of ease—and of likely-correct results—to move to the tidy divergent region of the functional equation, work there, and then take the answer back to the region of convergence.
5. The Bernoulli Numbers and Zeta Values by Contour Integration

First, for $\Re(s) > 1$ compute, using the invariance of the Haar measure $dt/t$ under $t \mapsto nt$ and the formula $\sum_{n \geq 1} r^n = 1/(r^{-1} - 1)$ if $|r| < 1$ for the second and fourth equalities to follow,

$$
\Gamma(s)\zeta(s) = \int_{t=0}^{\infty} e^{-ts} \frac{dt}{t} \cdot \sum_{n \geq 1} n^{-s} = \sum_{n \geq 1} n^{-s} \int_{t=0}^{\infty} e^{-nt} (nt)^s \frac{dt}{t} = \int_{t=0}^{\infty} \sum_{n \geq 1} e^{-nt} t^s \frac{dt}{t} = \int_{t=0}^{\infty} t^s \frac{dt}{e^t - 1}.
$$

That is,

$$
\Gamma(s)\zeta(s) = \int_{t=0}^{\infty} t^s \frac{dt}{e^t - 1}, \quad \Re(s) > 1.
$$

Here the condition $\Re(s) > 1$ is required for the integral to converge at its left endpoint, but it converges for all complex $s$ at its right end. Now consider the complex contour integral

$$
\int_{H_s} \frac{z^s}{e^z - 1} \frac{dz}{z}
$$

where $H_s$, the Hankel contour or keyhole contour, traverses the bottom side of the positive real axis from $+\infty$ in to some small $\varepsilon$, then a clockwise circle of radius $\varepsilon$ about 0, then the top side of the positive real axis from $\varepsilon$ back out to $\infty$. Because $z$ has argument $2\pi$ on the inward portion of the contour and argument 0 on the outward portion, and $z^s = |z|^s e^{i \arg(z)s}$, while the integrand is small on the circle, the limiting value of this integral is

$$
\lim_{\varepsilon \to 0} \int_{H_s} \frac{z^s}{e^z - 1} \frac{dz}{z} = (1 - e^{2\pi is}) \int_{t=0}^{\infty} \frac{t^s}{e^t - 1} \frac{dt}{t} = (1 - e^{2\pi is})\Gamma(s)\zeta(s).
$$

That is,

$$
\zeta(s) = \frac{1}{\Gamma(s)(1 - e^{2\pi is})} \lim_{\varepsilon \to 0} \int_{H_s} \frac{z^s}{e^z - 1} \frac{dz}{z}, \quad \Re(s) > 1.
$$

We show that this formula also gives values for $\zeta(s)$ where $s$ is a nonpositive integer.

For $s \sim 1 - k$ with $k \in \{1, 2, 3, \ldots\}$ we have on the right side of the previous display, noting that $e^{2\pi is} = e^{2\pi i(s+k-1)}$ with $s+k-1$ small,

$$
\Gamma(s) \sim \frac{(-1)^{k-1}}{(k-1)!(s+k-1)} \quad \text{and} \quad 1 - e^{2\pi is} \sim -2\pi i(s+k-1),
$$

so that, again making a limit tacit,

$$
\frac{1}{\Gamma(1-k)(1 - e^{2\pi i(1-k)})} = \frac{(-1)^{k-1}(k-1)!}{-2\pi i}.
$$

Also, for $s = 1 - k$ the inward and outward portions of $\int_{H_s} \frac{z^s}{e^z - 1} \frac{dz}{z}$ cancel because for $x > 0$ the quantities $(xe^{2\pi i})^s = x^s e^{i \arg(z)s}$ and $x^s$ are equal. The integral over the clockwise circle around 0 gives $-2\pi i \text{Res}_0(f)$ where $f(z) = z^{-k}/(e^z - 1)$, and because

$$
\frac{z^{-k}}{e^z - 1} = \frac{z^{-k-1}}{e^z} = z^{-k-1} \sum_{\ell \geq 0} B_{\ell} z^\ell
$$

we have

$$
\zeta(1-k) = \frac{1}{\Gamma(1-k)} \frac{1}{1 - e^{2\pi i}} = \frac{1}{\Gamma(1-k)} \frac{1}{1 - e^{2\pi i(1-k)}}.
$$
the residue at 0 is $B_k/k!$. Assemble these observations to obtain a value for $\zeta(1-k)$,

$$\zeta(1-k) = \frac{(-1)^{k-1}(k-1)!}{-2\pi i} \frac{B_k}{k!} = (-1)^{k-1} \frac{B_k}{k}.$$  

For $k \geq 2$ even, this reproduces the formula $\zeta(1-k) = -B_k/k$ from the previous section. For $k \geq 3$ odd, this gives $\zeta(-2) = \zeta(-4) = \cdots = 0$. For $k = 1$, this gives $\zeta(0) = -1/2$. This method doesn’t rely on having $\zeta(k)$ for even $k \geq 2$ first.