MATHEMATICS 332: ALGEBRA — SOLUTION TO RIGHT-ADJOINT EXERCISE

Let A and B be rings-with-1, and let $\alpha : A \longrightarrow B$ be a ring homomorphism such that $\alpha(1_A) = 1_B$. The hom-group functor from A-modules to B-modules is

$$N \longmapsto \mathrm{Hom}_A(B, N),$$

where the B-module structure of $\mathrm{Hom}_A(B, N)$ is that for $b \in B$, $f \in \mathrm{Hom}_A(B, N)$,

$$(bf)(x) = f(xb) \quad \text{for all } x \in B.$$

For maps, the functor is composition. That is, if $g : N \longrightarrow N'$ is an A-module map then its induced map is postcomposition,

$$g \circ - : \mathrm{Hom}_A(B, N) \longrightarrow \mathrm{Hom}_A(B, N'), \quad f \longmapsto g \circ f.$$

Prove that hom-group formation is a right-adjoint of restriction, is natural in M, and is natural in N.

Proof. For the right-adjointness, define

$$i_{M,N} : \mathrm{Hom}_B(M, \mathrm{Hom}_A(B, N)) \longrightarrow \mathrm{Hom}_A(\mathrm{Res}_A^B M, N)$$

by the formula

$$(i_{M,N}\Phi)(m) = (\Phi(m))(1_B), \quad m \in M,$$

and define

$$j_{M,N} : \mathrm{Hom}_A(\mathrm{Res}_A^B M, N) \longrightarrow \mathrm{Hom}_B(M, \mathrm{Hom}_A(B, N))$$

by the formula

$$(j_{M,N}\phi)(m) = \big(b \longmapsto \phi(bm)\big), \quad b \in B,\ m \in M.$$

Then i and j are readily seen to be abelian group homomorphisms, and (all the symbols meaning what they must)

$$\begin{aligned}
(ji\Phi)(m) &= \big(b \longmapsto \boxed{}\big) && \text{by definition of } j \\
&= \big(b \longmapsto \boxed{}\big) && \text{by definition of } i \\
&= \big(b \longmapsto \boxed{}\big) && \text{since } \Phi \text{ is } B\text{-linear} \\
&= \big(b \longmapsto \boxed{}\big) && \text{by the } B\text{-module structure of } \mathrm{Hom}_A(B, N) \\
&= \big(b \longmapsto (\Phi(m))(b)\big),
\end{aligned}$$

showing that $(ji\Phi)(m) = \Phi(m)$ for all m, i.e., $ji\Phi = \Phi$. Also,

$$\begin{aligned}
(ij\phi)(m) &= \boxed{} && \text{by definition of } i \\
&= \big(b \longmapsto \boxed{}\big)(\boxed{}) && \text{by definition of } j \\
&= \phi(m),
\end{aligned}$$

showing that $ij\phi = \phi$. Thus $i_{M,N}$ is an isomorphism.

Naturality in M means that for every B-module map $f : M' \longrightarrow M$ and every A-module N, the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{Hom}_B(M, \mathrm{Hom}_A(B, N)) & \xrightarrow{i_{M,N}} & \mathrm{Hom}_A(\mathrm{Res}^B_A M, N) \\
\Big\downarrow {\scriptstyle -\circ f} & & \Big\downarrow {\scriptstyle -\circ \mathrm{Res}^B_A f} \\
\mathrm{Hom}_B(M', \mathrm{Hom}_A(B, N)) & \xrightarrow{i_{M',N}} & \mathrm{Hom}_A(\mathrm{Res}^B_A M', N).
\end{array}$$

Consider any B-linear $\Phi : M \longrightarrow \mathrm{Hom}_A(B, N)$. Taking it across the top of the diagram gives

$$i_{M,N}\Phi : m \longmapsto \boxed{},$$

and taking this down the right side of the diagram gives in turn

$$i_{M,N}\Phi \circ \mathrm{Res}^B_A f : m' \longmapsto \boxed{}.$$

On the other hand, taking Φ down the left side of the diagram gives $\Phi \circ f$, which is taken across the bottom of the diagram to the same thing as a moment ago but with the symbols regrouped,

$$i_{M',N}(\Phi \circ f) : m' \longmapsto \boxed{}.$$

Finally, naturality in N means that for every B-module M and every A-module map $g : N \longrightarrow N'$, the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{Hom}_B(M, \mathrm{Hom}_A(B, N)) & \xrightarrow{i_{M,N}} & \mathrm{Hom}_A(\mathrm{Res}^B_A M, N) \\
{\scriptstyle g\circ -} \Big\downarrow & & \Big\downarrow {\scriptstyle \mathrm{Res}^B_A g \circ -} \\
\mathrm{Hom}_B(M, \mathrm{Hom}_A(B, N')) & \xrightarrow{i_{M,N'}} & \mathrm{Hom}_A(\mathrm{Res}^B_A M, N').
\end{array}$$

Consider any B-linear $\Phi : M \longrightarrow \mathrm{Hom}_A(B, N)$. Taking it across the top of the diagram and then down the right side gives

$$\mathrm{Res}^B_A g \circ i_{M,N}\Phi : m \longmapsto \boxed{}.$$

On the other hand, taking it down the left side of the diagram and across the bottom gives the same thing but with the symbols regrouped,

$$i_{M,N'}(g \circ \Phi) : m \longmapsto \boxed{}.$$

$\square$