MATHEMATICS 332: ALGEBRA — EXERCISES ON DISCRIMINANTS AND RESULTANTS

Reading: Gallian, chapter 16 and 17; class handout on polynomials.

Problems:

1. Let r_1, r_2, r_3 be algebraically related by $r_1 + r_2 + r_3 = 0$. Express the discriminant Δ in terms of $\sigma_1, \sigma_2, \sigma_3$. (Do not quote the more general result from the handout, of course.)

2. Show that a rational function

$$\frac{f(r_1, \cdots, r_n)}{g(r_1, \cdots, r_n)} \quad \text{where } f, g \in \mathbb{Z}[X_1, \cdots, X_n]$$

is S_n-invariant if and only if it takes the form

$$\frac{h(\sigma_1, \cdots, \sigma_n)}{k(\sigma_1, \cdots, \sigma_n)} \quad \text{where } h, k \in \mathbb{Z}[X_1, \cdots, X_n].$$

3. (a) Establish the formula for the **Vandermonde determinant**,

$$\begin{vmatrix} 1 & r_1 & r_1^2 & \cdots & r_1^{n-1} \\ 1 & r_2 & r_2^2 & \cdots & r_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & r_n & r_n^2 & \cdots & r_n^{n-1} \end{vmatrix} = \prod_{i<j} (r_j - r_i).$$

(Replace the last column by $(p(r_1), \cdots, p(r_n))$ where $p(X) = \prod_{i=1}^{n-1}(X - r_i)$.) Left-multiply the Vandermonde matrix by its transpose and take determinants to obtain

$$\begin{vmatrix} s_0 & s_1 & \cdots & s_{n-1} \\ s_1 & s_2 & \cdots & s_n \\ \vdots & \vdots & & \vdots \\ s_{n-1} & s_n & \cdots & s_{2n-2} \end{vmatrix} = \Delta(r_1, \cdots, r_n).$$

This equality expresses the discriminant in terms of the elementary symmetric functions $\sigma_1, \cdots, \sigma_n$ since Newton's identities give expressions for the power sums s_j in terms of the σ_j.

(b) Show that the determinant of the n-by-n derivative matrix of the elementary symmetric functions $\sigma_i(r_1, \cdots, r_n)$ is the square root of the discriminant of the indeterminates,

$$\det[D_j \sigma_i] = \prod_{i<j} (r_j - r_i).$$

Show also that the jth partial derivative of the ith elementary symmetric function is

$$D_j \sigma_i = \sigma_{i-1}(r_1, \cdots, \overline{r_j}, \cdots, r_n),$$

where the overbar means the variable is not present.

4. (a) Show that p and q share a nonconstant factor in $k[X]$ if and only if there exist nonzero polynomials P of degree less than n and Q of degree less than m in $k[X]$ such that $pP = qQ$.

(b) Write out the Sylvester matrix M for various small values of m and n, and compute the corresponding resultants.

(c) Fill in the details of the proof of Theorem 5.2.

5. (a) Use Theorem 5.2 to show that if $f(X)$ is monic, so that

$$f'(X) = \sum_{i=1}^{n} \prod_{j \neq i} (X - r_j),$$

then

$$R(f, f') = (-1)^{n(n-1)/2} \Delta(f).$$

(Note: The argument should use only the algebraic operations, sum and product, not division or the square root.)

(b) Use part (a) and the determinantal definition of the resultant to recompute the discriminants of $f(X) = X^2 + bX + c$ and of $f(X) = X^3 + bX + c$.

6. (a) Lightly prove the formulas in Corollary 5.3.

(b) Let $f(X) = X^n + bX + c$. Compute $\Delta(f) = (-1)^{n(n-1)/2} R(f, f')$ using the corollary. (Do a polynomial division and apply the second formula in (3). The answer is $(-1)^{(n-1)(n-2)/2}(n-1)^{n-1}b^n + (-1)^{n(n-1)/2} n^n c^{n-1}$, and you can check that it generalizes the answers for $n = 2$ and $n = 3$ from the previous exercise.) Since n is a general symbol here, this method is much tidier than trying to evaluate $R(f, f')$ as a variably-sized large determinant.