ABELIANIZING A GROUP

Let G be a group. Its centralizer, $Z(G)$, is an abelian normal subgroup. We also would like an abelian *quotient* of G that retains as much information about G as possible.

The *commutator* of any two elements a and b of G is defined as

$$[a, b] = aba^{-1}b^{-1}.$$

The significance of the commutator is that

$$a \text{ and } b \text{ commute if and only if } [a, b] = e.$$

More specifically, note the formula

$$ab = [a, b]\, ba.$$

The *commutator subgroup* of G, or *first derived subgroup* of G, is the subgroup generated by the commutators,

$$[G, G] = \langle \{[a, b] : a, b \in G\} \rangle.$$

Thus a quotient G/K is abelian if and only if K contains $[G, G]$. Especially, we will show that $[G, G] \lhd G$, so that consequently:

> *The quotient $G/[G, G]$ is abelian, and every abelian quotient of G is a quotient of $G/[G, G]$.*

Proof. As explained, it suffices to show that $[G, G] \lhd G$, and thus it suffices to show that every commutator $[a, b]$ is conjugated by G back into $[G, G]$. But note that for any $a, b, c \in G$,

$$c[a, b]c^{-1} = [c, [a, b]]\, [a, b] \in [G, G].$$

□