BEGINNINGS OF COMPACT RIEMANN SURFACE THEORY

These notes sketch how Siegel space and the symplectic group arise naturally in the context of compact Riemann surfaces, and how the Jacobi theta function creates the prime form to construct meromorphic functions and to help prove the Abel–Jacobi Theorem.

1. Siegel space

Topologically, a compact Riemann surface X of genus g is a g-holed torus, i.e., a sphere with g handles. Give it a *marking* consisting of a *base point* P_0 and a *canonical homology basis* $A_1, \dots, A_g, B_1, \dots, B_g$. *Canonical* means that the intersection numbers of the paths are

$$A_i \cdot A_j = B_i \cdot B_j = 0, \qquad A_i \cdot B_j = \delta_{i,j} = -B_i \cdot A_j.$$

(If two oriented paths cross with the direction from the first to the second being counterclockwise then the intersection number is 1.) Arrayed, these relations produce the *skew matrix*, i.e.,

$$\begin{bmatrix} A_1 \\ \vdots \\ A_g \\ B_1 \\ \vdots \\ B_g \end{bmatrix} \cdot \begin{bmatrix} A_1 & \cdots & A_g & B_1 & \cdots & B_g \end{bmatrix} = \begin{bmatrix} 0_g & 1_g \\ -1_g & 0_g \end{bmatrix} \overset{\text{call}}{=} J.$$

The space of holomorphic 1-forms on X has dimension g as a complex vector space. The dimension is nonobvious, proved most intuitively in an old book by Klein using fluid flows and cut-and-paste techniques. Pick an ordered basis $\{\omega_1, \dots, \omega_g\}$.

The *period matrix* for X—as marked—is

$$U = \left[\int_{\begin{bmatrix} A \\ B \end{bmatrix}} \vec{\omega} \right]_{2g \times g}$$

where $\begin{bmatrix} A \\ B \end{bmatrix}$ is the homology basis column vector from above and $\vec{\omega}$ is the row vector $\vec{\omega} = (\omega_1, \dots, \omega_g)$. Thus $U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$ where $[U_1]_{i,j} = \int_{A_i} \omega_j$ and $[U_2]_{i,j} = \int_{B_i} \omega_j$.

Riemann's Bilinear Relations, assembled into matrix form, are

$$U^t J U = 0 \qquad \text{and} \qquad (1/i) \cdot \overline{U}^t J U > 0.$$

The relations also show that U_1 is invertible, so U can be normalized: right-multiply by U_1^{-1}, or, equivalently, pick a new basis $\vec{\omega}$ of 1-forms such that $\int_{A_i} \omega_j = \delta_{i,j}$. The normalized period matrix is

$$UU_1^{-1} = \begin{bmatrix} 1 \\ \Omega \end{bmatrix},$$

and Riemann's Bilinear Relations become, in this case,

$$\Omega^t = \Omega \qquad \text{and} \qquad \mathrm{Im}(\Omega) > 0.$$

That is, $\Omega \in \mathcal{H}_g$, where $\mathcal{H}_g$ is *Siegel upper half-space of dimension g*. So indeed Siegel space arises naturally in the Riemann surface context.

2. The symplectic group

Next, consider a change of canonical homology bases. Let a new basis be

$$\begin{bmatrix} A' \\ B' \end{bmatrix} = M \begin{bmatrix} A \\ B \end{bmatrix}, \qquad M \in \mathrm{GL}_{2g}(\mathbb{Z}).$$

The new basis is again canonical exactly when $\begin{bmatrix} A' \\ B' \end{bmatrix} \cdot \begin{bmatrix} A' & B' \end{bmatrix} = J$, i.e., $M \begin{bmatrix} A \\ B \end{bmatrix} \cdot \begin{bmatrix} A & B \end{bmatrix} M^t = J$, i.e., $MJM^t = J$. This is the defining condition for the *symplectic group* $\mathrm{Sp}_g(\mathbb{Z})$, equivalent—though not quite trivially—to its usual form,

$$M^t JM = J.$$

Thus, the change of canonical homology basis group is $\mathrm{Sp}_g(\mathbb{Z})$ in a natural way.

Assembling the results so far also shows how the symplectic group $\mathrm{Sp}_g(\mathbb{Z})$ acts on Siegel space $\mathcal{H}_g$. Recovering the familiar formula in the 1-dimensional case, where $\mathrm{SL}_2(\mathbb{Z})$ acts on the complex upper half-plane, requires compensating for the notational inconsistency that in one variable, vectors $\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ have their bottom coordinates normalized to 1, while period matrices are normalized up top. The adjustment is to compose with the outer automorphism $M \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} M \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ of $\mathrm{Sp}_g(\mathbb{Z})$. That is, the action of M on period matrices is defined as

$$\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \overset{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}{\longmapsto} \begin{bmatrix} U_2 \\ U_1 \end{bmatrix} \overset{M}{\longmapsto} \begin{bmatrix} U_2' \\ U_1' \end{bmatrix} \overset{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}{\longmapsto} \begin{bmatrix} U_1' \\ U_2' \end{bmatrix}.$$

If the symplectic matrix M has block structure $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ and the normalized period matrices are $\Omega = U_2 U_1^{-1}$ and $\Omega' = U_2' U_1'^{-1}$, then the relation between them now works out to the familiar

$$\Omega' = (A\Omega + B)(C\Omega + D)^{-1}.$$

3. The Jacobian

The period matrix Ω has an associated *lattice*

$$L = L_\Omega = \mathbb{Z}^g + \Omega\mathbb{Z}^g \subset \mathbb{C}^g.$$

The complex torus quotient $\mathbb{C}^g/L$ is the *Jacobian of X*, $\mathrm{Jac}(X)$. The Jacobian inherits a natural abelian group structure from $\mathbb{C}^g$.

The map from the Riemann surface to its Jacobian,

$$I : X \longrightarrow \mathrm{Jac}(X), \qquad P \mapsto \int_{P_0}^{P} \vec{\omega},$$

(recall that P_0 is the base point of the marking) is well defined, as changing the path of integration alters the integral by a period. It extends to a map from the degree-0 divisor group on X,

$$I : D_0 \longrightarrow \mathrm{Jac}(X), \qquad \sum n_i P_i \mapsto \sum n_i \int_{P_0}^{P_i} \vec{\omega}.$$

An easy complex analysis argument shows that $I(D_\ell) = 0$, where D_ℓ is the divisor subgroup linearly equivalent to 0, i.e., the divisors of meromorphic functions on X. *Abel's Theorem* says that I induces a group isomorphism,

$$I : D_0/D_\ell \xrightarrow{\sim} \mathrm{Jac}(X).$$

Among other things, this shows which meromorphic functions exist on X: only those whose divisors map to zero in the Jacobian under I. When $g = 0$ (recall that g is the genus), $\mathrm{Jac}(X) = \{0\}$, and Abel's Theorem gives the well known fact that the divisor class group on the Riemann sphere is trivial. When $g = 1$, the Jacobian is $\mathrm{Jac}(X) = X$ and I is the identity map, so Abel's Theorem gives the familiar constraint on the possible zeros and poles of a meromophic function on the torus.

4. Theta functions and Abel's Theorem

Let $\mathbf{e}(x) = e^{2\pi i x}$ for $x \in \mathbb{C}$. I will prove part of Abel's Theorem using the Jacobi theta function,

$$\vartheta(z, \Omega) = \sum_{n \in \mathbb{Z}^g} \mathbf{e}(\frac{1}{2} n^t \Omega n + n^t z), \qquad z \in \mathbb{C}^g, \Omega \in \mathcal{H}_g.$$

The theta series has absolute convergence properties justifying rearrangement. It is *quasi-periodic* with respect to the lattice L_Ω, meaning

$$\vartheta(z + m, \Omega) = \vartheta(z, \Omega), \qquad m \in \mathbb{Z}^g,$$

but

$$\vartheta(z + \Omega m, \Omega) = \vartheta(z, \Omega) \cdot \mathbf{e}(-\frac{1}{2} m^t \Omega m - m^t z), \qquad m \in \mathbb{Z}^g.$$

These are easy to show. For the first identity, note that $\frac{1}{2} n^t \Omega n + n^t(z + m)$ differs from the original exponent by an integer; for the second, compute

$$\frac{1}{2} n^t \Omega n + n^t(z + \Omega m) = \frac{1}{2}(n + m)^t \Omega (n + m) + (n + m)^t z - \frac{1}{2} m^t \Omega m - m^t z,$$

and the result follows by summing over $n + m$ rather than n.

Suppose that $\varepsilon \in \mathbb{C}^g$ satisfies $\vartheta(\varepsilon) = 0$. (From now on Ω is fixed, so ϑ will only receive one argument.) Define for any pair of points $x, y \in X$,

$$E_\varepsilon(x, y) = \vartheta(\varepsilon + \int_x^y \vec{\omega}).$$

This is the *prime form on* X. (The name "prime form" will become clear soon.) It is locally well-defined once points $x_0, y_0 \in X$ and a path from x_0 to y_0 are chosen and then the points are perturbed slightly. If moving the points around X back to themselves changes the path between them by an A_k, the prime form is left invariant since the integral changes by an integer vector. But if the path changes by a B_k, the integral changes by the kth column of Ω, i.e., by Ωe_k (where e_k is a standard basis vector). Quasi-periodicity shows that the effect is to multiply $E_\varepsilon(x, y)$ by the factor

$$\mathbf{e}(-\frac{1}{2}e_k^t\Omega e_k - e_k^t(\varepsilon + \int_x^y \vec{\omega})) = \mathbf{e}(-\frac{1}{2}\Omega_{k,k} - \varepsilon_k - \int_x^y \omega_k),$$

where the integral is along the original path.

Returning to Abel's Theorem, we are given a degree-0 divisor $\delta = \sum_{i=1}^d Z_i - \sum_{i=1}^d P_i$ such that $I(\delta) = 0$ in $\mathrm{Jac}(X)$, i.e., $\sum(\int_{P_0}^{Z_i} - \int_{P_0}^{P_i})\vec{\omega} \in L_\Omega$. Choose specific paths from P_0 to the Z_i and from P_0 to the P_i. Having done so, change one of the paths so that even in the full space $\mathbb{C}^g$,

$$\sum(\int_{P_0}^{Z_i} - \int_{P_0}^{P_i})\vec{\omega} = 0.$$

The goal is to find a meromorphic function f on X with zeros Z_i and poles P_i.

Such a function is

$$f(y) = \frac{\prod_{i=1}^d E_\varepsilon(Z_i, y)}{\prod_{i=1}^d E_\varepsilon(P_i, y)}.$$

This clearly has zeros at the Z_i and poles at the P_i, and by some technicalities that I am skipping, ε can be chosen so that f has no other zeros or poles.

To ensure that f is well defined, make the convention that in $E_\varepsilon(Z_i, y) = \vartheta(\varepsilon + \int_{Z_i}^y \vec{\omega})$, the path of integration proceeds from Z_i to P_0 as specified a moment ago, and then from P_0 on to y. Similarly for $E_\varepsilon(P_i, y)$, using the same path from P_0 to y. Under this convention, $f(y)$ depends only on the choice of path from P_0 to y. As before, altering this path by an A_k leaves each term of $f(y)$ invariant. And altering by a B_k multiplies $f(y)$ by a net factor of

$$\mathbf{e}(\sum(-\frac{1}{2}\Omega_{k,k} - \varepsilon_k - \int_{Z_i}^y \omega_k + \frac{1}{2}\Omega_{k,k} + \varepsilon_k + \int_{P_i}^y \omega_k))$$

which simplifies to

$$\mathbf{e}(\sum(\int_{P_0}^{Z_i} - \int_{P_0}^{P_i})\omega_k) = \mathbf{e}(0) = 1,$$

by how the paths from P_0 to the Z_i and the P_i were specified. So indeed f makes sense and has the required divisor.

The function $E_\varepsilon(x, y)$ is called the prime form on X because, as just shown, it is the basic building block for all meromorphic functions. By analogy, the function $E(x, y) = x - y$ is the prime form on the Riemann sphere since any meromorphic function there takes the form

$$f(y) = c\frac{\prod E(Z_i, y)}{\prod E(P_i, y)}.$$