TERMWISE DERIVATIVES OF POWER SERIES

This writeup gives two brief proofs that a power series can be differentiated termwise within its disk of convergence. The first proof builds on work that we have already done, and the second uses the Fundamental Theorem of Calculus. Consider a power series

$$p(z) = \sum_{j=0}^{\infty} a_j (z-c)^j = \lim_{n\to\infty} p_n(z),$$

where its partial sums are

$$p_n(z) = \sum_{j=0}^{n} a_j (z-c)^j, \quad n = 0, 1, 2, \dots.$$

Assume that p has a positive radius of convergence, and let D denote its disk of convergence. We show that p is differentiable at any $z \in D$, and

$$p'(z) = \lim_{n\to\infty} p_n'(z) = \sum_{j=1}^{\infty} j a_j (z-c)^{j-1}.$$

1. First argument: Justify a limit-exchange by using two others

In previous work, we have

passed a z-derivative through a ζ-integral

to establish Cauchy's integral representation formula for derivatives, and we have

passed an n-limit of sums through a ζ-integral

to establish the power series representation of complex-differentiable functions. To establish the differentiability of complex power series, we need to

pass a z-derivative through an n-limit of sums.

This limit-exchange reduces to the two limit-exchanges that we have already carried out.

Let $p(z)$ and D be as above. Let z be any point of D. Some closed disk $\overline{B}$ about z of positive radius lies in D. Since $\overline{B}$ is compact and p converge uniformly on $\overline{B}$, the restriction of p to $\overline{B}$ is continuous, and so p itself is continuous on the interior of $\overline{B}$; in particular p is continuous at z. Since z is arbitrary, p is continuous on all of D.

Again let z be any point of D. Let γ be a small circle around z in D. Then p is continuous and hence integrable on γ, and also p converges uniformly on γ.

The calculation proceeds as follows. By the definition of the infinite sum as the limit of the finite sums, because each finite sum is differentiable and therefore has integral representation, and because the integrand converges uniformly on γ, so that the limit passes through the integral,

$$p(z) = \lim_{n\to\infty} p_n(z) = \lim_{n\to\infty} \frac{1}{2\pi i}\int_\gamma \frac{p_n(\zeta)\,\mathrm{d}\zeta}{\zeta - z} = \frac{1}{2\pi i}\int_\gamma \frac{p(\zeta)\,\mathrm{d}\zeta}{\zeta - z}.$$

The power series is now part of the integrand, but it no longer involves z. The convergence

$$p(\zeta) \cdot \frac{\frac{1}{\zeta - z'} - \frac{1}{\zeta - z}}{z' - z} \xrightarrow{z' \to z} p(\zeta) \cdot \frac{1}{(\zeta - z)^2}$$

is uniform on γ, allowing us to take the z-derivative by passing it through the integral,

$$p'(z) = \frac{\mathrm{d}}{\mathrm{d}z} \left[\frac{1}{2\pi i} \int_\gamma \frac{p(\zeta)\,\mathrm{d}\zeta}{\zeta - z} \right] = \frac{1}{2\pi i} \int_\gamma \frac{\partial}{\partial z} \frac{p(\zeta)\,\mathrm{d}\zeta}{(\zeta - z)^2} = \frac{1}{2\pi i} \int_\gamma \frac{p(\zeta)\,\mathrm{d}\zeta}{(\zeta - z)^2}.$$

Pass the limit back out through the integral, and then use the integral representation of the derivative to get

$$p'(z) = \frac{1}{2\pi i} \int_\gamma \frac{p(\zeta)\,\mathrm{d}\zeta}{(\zeta - z)^2} = \lim_{n\to\infty} \frac{1}{2\pi i} \int_\gamma \frac{p_n(\zeta)\,\mathrm{d}\zeta}{(\zeta - z)^2} = \lim_{n\to\infty} p_n'(z),$$

where again the limit passes through the integral because the integrand converges uniformly on γ, and where the last limit exists because the one before it exists.

2. Second argument: Use the Fundamental Theorem

Also, the termwise differentiability result can be proved by using the Fundamental Theorem of Calculus. Now take $a_0 = 0$ and $c = 0$ in our power series to lighten the notation. For any point z in D, the partial sums of the termwise derivative of the power series converge uniformly to the termwise derivative on the line segment from 0 to z. Integrate along this segment and use one version of the Fundamental Theorem of Calculus to get

$$p(z) = \lim_{n\to\infty} p_n(z) = \lim_{n\to\infty} \int_{\zeta=0}^{z} p_n'(\zeta)\,\mathrm{d}\zeta = \int_{\zeta=0}^{z} \lim_{n\to\infty} p_n'(\zeta)\,\mathrm{d}\zeta.$$

Thus the other version of the Fundamental Theorem says that $p'(z)$ exists and it is

$$p'(z) = \lim_{n\to\infty} p_n'(z).$$

That is, again, the power series can be differentiated termwise within its disk of convergence.