SIMPLE PROOF OF THE PRIME NUMBER THEOREM

This writeup is drawn from a writeup by Paul Garrett for his complex analysis course,


Especially, the bibliography of the source writeup contains relevant papers of Chebyshev, Erdős, Garrett, Hadamard, Newman, de la Vallée Poussin, Riemann, Selberg, and Wiener.

The prime-counting function, a function of a real variable, is

$$\pi(x) = |\{p : p \leq x\}|.$$ 

That is, \( \pi(x) \) equals the number of prime numbers that are at most \( x \). The **Prime Number Theorem** states that

\[ \pi(x) \sim \frac{x}{\log(x)} \]

meaning that \( \lim_{x \to \infty} \pi(x)/(x/\log(x)) = 1 \).

The Chebyshev theta function, also a function of a real variable, is

$$\Theta(x) = \sum_{p \leq x} \log p.$$ 

A quick argument shows that \( \Theta(x) = O(x) \), meaning that \( \Theta(x) \leq cx \) for some \( c \) and all large \( x \); in fact, the argument produces such a \( c \) and the inequality holds for all \( x \). A basic lemma of asymptotics specializes to show that if \( \Theta(x) \sim x \), meaning that \( \lim_{x \to \infty} \Theta(x)/x = 1 \), then \( \pi(x) \sim x/\log x \), giving the Prime Number Theorem. Thus the main work of this writeup is to go from \( \Theta(x) = O(x) \) to \( \Theta(x) \sim x \). With \( \zeta(s) \) the Euler–Riemann zeta function, the dominant term of \( \zeta'(s)/\zeta(s) \) near \( s = 1 \) is a Dirichlet-like series closely related to \( \Theta(x) \). This fact combines with the convergence theorem in section 4 below to finish the proof.

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1. Weak theta asymptotic

A quick argument shows that
\[ \vartheta(x) = O(x) \]
as follows. For any positive integer \( n \),
\[ \prod_{n < p \leq 2n} p \leq \binom{2n}{n} < \sum_{j=0}^{2n} \binom{2n}{j} = 2^{2n}, \]
and so
\[ \vartheta(2n) - \vartheta(n) = \sum_{n < p \leq 2n} \log p = \log \left( \prod_{n < p \leq 2n} p \right) < 2n \log 2. \]
It follows that
\[ \vartheta(2^m) < 2^{m+1} \log 2, \quad m \in \mathbb{Z}_{\geq 1}, \]
and now, because any \( x > 1 \) satisfies \( 2^{m-1} < x \leq 2^m \) for some such \( m \),
\[ \vartheta(x) < 2^{m+1} \log 2 < 4x \log 2. \]
So indeed \( \vartheta(x) = O(x) \).

2. Lemma on asymptotics, beginning of the proof

2.1. Lemma. The following lemma is elementary and ubiquitous in asymptotics.

Lemma 2.1. Suppose that a sequence \( \{ c_n \} \) satisfies
\[ \sum_{n \leq x} c_n \log n \sim rx \quad \text{for some } r. \]
Then
\[ \sum_{n \leq x} c_n \sim \frac{rx}{\log x}. \]

Proof. Name the two sums in the lemma,
\[ \theta(x) = \sum_{n \leq x} c_n \log n \quad \text{and} \quad \varphi(x) = \sum_{n \leq x} c_n. \]
Thus \( \theta(x) \sim rx \), and we want to show that \( \varphi(x) \sim rx / \log x \). Because the step function \( \theta(x) \) jumps by \( c_n \log n \) at each \( n \), and the step function \( \varphi(x) \) jumps by \( c_n \) at each \( n \), we have for \( t > 1 \) in the sense of Stieltjes integration,
\[ d\varphi(t) = \frac{d\theta(t)}{\log t}. \]
With “∗” denoting a fixed, large enough lower limit of integration, and with a Stieltjes integral and integration by parts,

\[ \varphi(x) \sim \int_{t=x}^{x} dt \frac{\theta(t)}{\log t} = \frac{\theta(x)}{\log x} \bigg|_{t=x} + \int_{t=x}^{x} \frac{\theta(t)}{t \log^2 t} dt. \]

The boundary term is asymptotically \( rx/\log x \), as desired for \( \varphi(x) \), so what needs to be shown is that the last integral in (1) is \( o(x/\log x) \).

Because \( \frac{\theta(t)}{t} \sim r \) for large \( t \), estimate the integral of \( \frac{1}{\log 2t} \), first breaking it into two pieces,

\[ \int_{t=x}^{x} \frac{1}{\log^2 t} dt = \int_{t=x}^{\sqrt{x}} \frac{1}{\log^2 t} dt + \int_{t=\sqrt{x}}^{x} \frac{1}{\log^2 t} dt. \]

For the first piece,

\[ \int_{t=x}^{\sqrt{x}} \frac{1}{\log^2 t} dt \leq \sqrt{x} \int_{t=x}^{\sqrt{x}} \frac{1}{t \log^2 t} dt = -\sqrt{x} \frac{1}{\log t} \bigg|_{t=x}^{\sqrt{x}} \sim \sqrt{x}, \]

while for the second,

\[ \int_{t=\sqrt{x}}^{x} \frac{1}{\log^2 t} dt \leq \frac{1}{\log^2 \sqrt{x}} (x - \sqrt{x}) \sim \frac{2x}{\log^2 x}. \]

Altogether \( \int_{t=x}^{x} dt/\log^2 t \) is \( o(x/\log x) \). Because \( \theta(t)/t = O(1) \), the last integral in (1) is therefore \( o(x/\log x) \) as well, and the argument is complete.

\[ \square \]

2.2. **Beginning of the proof.** Consider the sequence \( \{c_n\} = \{c_1, c_2, \ldots\} \) where

\[ c_n = \begin{cases} 1 & \text{if } n \text{ is prime} \\ 0 & \text{otherwise} \end{cases} \]

The Chebyshev theta function and the prime-counting function are naturally re-expressed using this sequence,

\[ \vartheta(x) = \sum_{n \leq x} c_n \log n \quad \text{and} \quad \pi(x) = \sum_{n \leq x} c_n. \]

Consequently the lemma reduces the Prime Number Theorem to showing that

\[ \vartheta(x) \sim x \]

Already \( \vartheta(x) = O(x) \) is established, so the work is to go from this to the boxed result.

3. **Euler–Riemann zeta function**

3.1. **Zeta as a sum.** The Euler–Riemann zeta function is initially defined as a sum on an open right half plane,

\[ \zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \quad \text{Re}(s) > 1. \]

This sum converges absolutely on \( \text{Re}(s) > 1 \) because \( |n^{-s}| = n^{-\text{Re}(s)} \), and hence it indeed converges on \( \text{Re}(s) > 1 \). Each truncation \( \sum_{n=1}^{N} n^{-s} \) of the sum is entire.
Let $K$ denote a compact subset of $\text{Re}(s) > 1$. There exists some $\sigma > 1$ such that $\text{Re}(s) \geq \sigma$ on $K$, and so
\[
\left| \sum_{n=N}^{\infty} n^{-s} \right| \leq \sum_{n=N}^{\infty} n^{-\sigma}, \quad s \in K.
\]

This shows that the sum $\zeta(s)$ converges uniformly on $K$. Altogether, $\zeta(s)$ is holomorphic on $\text{Re}(s) > 1$.

3.2. **Zeta as a product.** The Euler–Riemann zeta function has a second expression as a product of so-called Euler factors over the prime numbers,
\[
\zeta(s) = \prod_p (1 - p^{-s})^{-1}, \quad \text{Re}(s) > 1.
\]
The equality of the product and sum expressions of $\zeta(s)$ for $\text{Re}(s) > 1$ is a matter of the geometric series formula and the Fundamental Theorem of Arithmetic, as follows. Consider any positive integer $k$, let $p_1, \ldots, p_k$ denote the first $k$ primes, compute
\[
\prod_{i=1}^{k}(1 - p_i^{-s})^{-1} = \prod_{i=1}^{k} \lim_{M_i \to \infty} \sum_{m_i=0}^{M_i} p_i^{-m_is} = \lim_{M_1, \ldots, M_k \to \infty} \prod_{i=1}^{k} \sum_{m_i=0}^{M_i} p_i^{-m_is}
\]
\[
= \lim_{M_1, \ldots, M_k \to \infty} \sum_{n=\prod_{i=1}^{k} p_i^{m_i}, m_i \leq M_i} n^{-s} = \sum_{n=\prod_{i=1}^{k} p_i^{m_i}} n^{-s},
\]
and take the limit as $k$ goes to $\infty$ to get the result, $\prod_p (1 - p^{-s})^{-1} = \sum_{n=1}^{\infty} n^{-s}$. Now the product form of $\zeta(s)$ inherits the holomorphy of the sum form.

Also we can show that the product is a holomorphic function on $\text{Re}(s) > 1$ with no reference to its matching the sum. Recall a general result for a product $\prod_{n=1}^{\infty}(1 + \varphi_n(s))$ with each $\varphi_n$ holomorphic on a domain $\Omega$, as follows. **Suppose that:**

For every compact $K$ in $\Omega$

there exists a summable sequence $\{x_n\} = \{x_n(K)\}$ of nonnegative real numbers such that

$|\varphi_n(s)| \leq x_n$ for all $n$, uniformly over $s \in K$.

Then $\prod_{n=1}^{\infty}(1 + \varphi_n(s))$ is holomorphic on $\Omega$. In our case, $\Omega$ is $\text{Re}(s) > 1$, and $\varphi_n(s)$ is $(1 - p^{-s})^{-1} - 1 = (1 - p^{-s})^{-1} p^{-s}$ if $n$ is a prime $p$, while $\varphi_n = 0$ if $n$ is composite. Let $K$ be a compact subset of $\text{Re}(s) > 1$. There exists some $\sigma > 1$ such that $\text{Re}(s) \geq \sigma$ on $K$. Let $\{x_n\} = \{2n^{-\sigma}\}$. For any prime $p$, for all $s \in K$,

$|\varphi_p(s)| = |(1 - p^{-s})^{-1} p^{-s}| \leq 2p^{-\sigma} = x_p$,

and $|\varphi_n(s)| = 0 \leq x_n$ for composite $n$ and $s \in K$. Thus the product $\prod_p (1 - p^{-s})^{-1}$ is holomorphic on $\text{Re}(s) > 1$, as claimed.

3.3. **Euler’s proof.** Using the product form of $\zeta(s)$, consider the logarithm of the zeta function for $s$ approaching 1 from the right,
\[
(2) \quad \log \zeta(s) = \sum_p \log((1 - p^{-s})^{-1}) = \sum_p \sum_{m \geq 1} \frac{1}{mp^{ms}}.
\]
This decomposes into two terms,

$$\log \zeta(s) = \sum_p \frac{1}{p^s} + \sum_p \sum_{m \geq 2} \frac{1}{m p^{ms}}.$$

The sum form of $\zeta(s)$ shows that $\zeta$ diverges at 1, and hence so does $\log \zeta$ although more slowly. The second sum is readily bounded by 1,

$$\sum_p \sum_{m \geq 2} \frac{1}{m p^{ms}} < \sum_p \frac{1}{p^s(1 - p^{-s})} = \sum_p \frac{1}{p^s(p^s - 1)} < \sum_{k=2}^{\infty} \frac{1}{k(k - 1)} = 1.$$

So the first sum $\sum_p p^{-s}$ is asymptotic to $\log \zeta(s)$ as $s$ goes to 1, and consequently the prime numbers are dense enough to make the sum diverge at $s = 1$. This is a stronger result than the existence of infinitely many primes. For the Prime Number Theorem, we will similarly study $(\log \zeta(s))' = \zeta'(s)/\zeta(s)$ at $s = 1$.

### 3.4. Continuation of zeta and its logarithmic derivative.

The function $\zeta(s)$ continues meromorphically to $\text{Re}(s) > 0$, the only singularity of the extension being a simple pole at $s = 1$ with residue $\text{res}_1 \zeta(s) = 1$. The argument requires some estimation but isn’t deep, as follows. For $\text{Re}(s) > 1$, introduce the function

$$\psi(x) = \zeta(s) - \frac{1}{s-1} = \sum_{n=1}^{\infty} n^{-s} - \int_1^{\infty} t^{-s} \, dt = \sum_{n=1}^{\infty} \int_n^{n+1} (n^{-s} - t^{-s}) \, dt.$$

This last sum is an infinite sum of analytic functions. For positive real $s$ it is the sum of small areas above the $y = t^{-s}$ curve but inside the circumscribing box of the curve over each unit interval, and hence it is bounded absolutely by 1. More generally, for complex $s$ with positive real part we can quantify the smallness of the sum as follows. For all $t \in [n, n+1]$ we have

$$|n^{-s} - t^{-s}| = |s| \int_n^t x^{-s-1} \, dx \leq |s| \int_n^t x^{-\text{Re}(s)-1} \, dx \leq |s| n^{-\text{Re}(s)-1},$$

with the last quantity in the previous display independent of $t$ and having the power of $n$ smaller by 1. It follows that

$$\left| \int_n^{n+1} (n^{-s} - t^{-s}) \, dt \right| \leq |s| n^{-\text{Re}(s)-1}.$$

This estimate shows that the sum $\psi(s)$ converges on $\{s : \text{Re}(s) > 0\}$, uniformly on compact subsets, making $\psi(s)$ analytic there. Thus, in the relation

$$\zeta(s) = \psi(s) + \frac{1}{s-1}, \quad \text{Re}(s) > 1,$$

the right side is meromorphic on $\text{Re}(s) > 0$, its only singularity being a simple pole at $s = 1$ with residue 1. So the previous display extends $\zeta(s)$ to $\text{Re}(s) > 0$ and gives it the same properties, as claimed.

The value $\psi(1) = \lim_{s \to 1} (\zeta(s) - \frac{1}{s-1})$ is called Euler’s constant and denoted $\gamma$,

$$\zeta(s) = \frac{1}{s-1} + \gamma + O(s-1), \quad \gamma = \sum_{n=1}^{\infty} \int_n^{n+1} (n^{-1} - t^{-1}) \, dt.$$

With $H_N$ denoting the $N$th harmonic number $\sum_{n=1}^{N} n^{-1}$, Euler’s constant is

$$\gamma = \lim_{N \to \infty} (H_N - \log N).$$
As above, this is the area above the $y = 1/x$ curve for $x \geq 1$ but inside the circumscribing boxes $[n, n+1] \times [0, 1/n]$ for $n \geq 1$.

The continuation argument just given should be viewed as a place-holder, because Riemann’s deeper argument continues $\zeta(s)$ meromorphically to all of the complex plane and establishes a functional equation for the continuation.

With $\zeta(s)$ continued, its logarithmic derivative $\zeta'(s)/\zeta(s)$ also continues meromorphically to $\text{Re}(s) > 0$, again having a simple pole at $s = 1$, this time with residue $\text{res}_1(\zeta'(s)/\zeta(s)) = \text{ord}_1 \zeta(s) = -1$. Indeed, recall more generally that if a function $f$ is meromorphic about $c$ and not identically $0$ then $f'/f$ is again meromorphic about $c$ with at most a simple pole at $c$, and

$$\text{res}_c(f'/f) = \text{ord}_c f.$$  

The argument is that because $f(z) = (z - c)^m g(z)$ about $c$, with $m = \text{ord}_c f$ and $g$ nonzero at $c$,

$$\frac{f'(z)}{f(z)} = \frac{m}{z - c} + \frac{g'(z)}{g(z)}, \quad \frac{g'}{g} \text{ holomorphic about } c,$$

and so $\text{res}_c(f'/f) = m$ as desired.

3.5. **Non-vanishing of zeta on** $\text{Re}(s) = 1$. To help prove the next proposition, and for further use in section 3.7, compute that for $\text{Re}(s) > 1$ the logarithmic derivative of $\zeta(s)$ is, from (2),

$$(3) \quad \frac{\zeta'(s)}{\zeta(s)} = (\log \zeta(s))' = -\sum_p \sum_{m \geq 1} \frac{\log p}{p^{ms}}.$$  

(The coefficient function of the Dirichlet series in the previous display is the von Mangoldt function, $\Lambda(p^m) = \log p$ and $\Lambda(n) = 0$ if $n$ is not a prime power.)  

**Proposition 3.1.** $\zeta(s) \neq 0$ for $\text{Re}(s) = 1$.

**Proof.** Fix any nonzero real $t$. Define

$$D(s) = \zeta(s)^3 \zeta(s + it)^4 \zeta(s + 2it).$$

From the logarithmic derivative computation just above,

$$\frac{D'(s)}{D(s)} = -\sum_p \sum_{m \geq 1} \frac{\log p(3 + 4p^{-mt} + p^{-2mt})}{p^{ms}}.$$  

We show that $0 \geq \text{ord}_1 D(s)$, i.e., $D(s)$ is nonzero at 1. The order of vanishing is

$$\text{ord}_1 D(s) = \text{res}_1(D'(s)/D(s)) = \lim_{s \to 1^+} (s - 1)D'(s)/D(s),$$

with $s$ approaching 1 from the right on the real axis. Because this quantity is an integer it is real, and so it is the limit of $s - 1$ times the real part of $D'(s)/D(s)$,

$$\text{ord}_1 D(s) = -\lim_{s \to 1^+} (s - 1) \sum_p \sum_{m \geq 1} \frac{(3 + 4 \cos(mt \log p) + \cos(2mt \log p)) \log p}{p^{ms}}.$$  

But for any real $\theta$,

$$3 + 4 \cos \theta + \cos 2\theta = 3 + 4 \cos \theta + 2 \cos^2 \theta - 1 = 2(1 + \cos \theta)^2 \geq 0,$$

and so the limit is nonpositive, i.e., $0 \geq \text{ord}_1 D(s)$ as claimed. The result follows because $\text{ord}_1 D(s) \geq -3 + 4 \text{ord}_1 \zeta(s + it)$, precluding the integer $\text{ord}_1 \zeta(s + it)$ from being positive. That is, $\zeta(1 + it) \neq 0$.  

$\square$
3.6. Improved continuation of the logarithmic derivative. In consequence of \( \zeta'(s)/\zeta(s) \) extending meromorphically from \( \text{Re}(s) > 1 \) to \( \text{Re}(s) > 0 \) with a simple pole at \( s = 1 \), and of \( \zeta(s) \) never vanishing on \( \text{Re}(s) = 1 \), also \( (s-1)\zeta'(s)/\zeta(s) \) extends holomorphically from \( \text{Re}(s) > 1 \) to \( \text{Re}(s) \geq 1 \). Being holomorphic on \( \text{Re}(s) \geq 1 \) and meromorphic on \( \text{Re}(s) > 0 \), \( (s-1)\zeta'(s)/\zeta(s) \) is in fact holomorphic on an open superset of \( \text{Re}(s) \geq 1 \).

3.7. Dominant term of the logarithmic derivative near \( s = 1 \). For \( \text{Re}(s) > 1 \), decompose the logarithmic derivative of \( \zeta(s) \) in (3) into two terms, as in Euler’s proof,

\[
\frac{\zeta'(s)}{\zeta(s)} = -\sum_p \log p \frac{1}{p^s} - \sum_p \sum_{m \geq 2} \log p \frac{1}{p^{ms}}, \quad \text{Re}(s) > 1.
\]

The second sum defines a holomorphic function on \( \text{Re}(s) > 1/2 \) because its partial sums are entire and it converges uniformly on compacta therein. Indeed, \( |p^{ms}| = p^{m\sigma} \) where \( \sigma = \text{Re}(s) \), and given \( \sigma > 1/2 \) there exists \( p_o \) such that \( \log p < p^{\sigma-1/2} \) for all \( p \geq p_o \); so, with \( c = 1/(1 - 2^{-1/2}) = 2 + \sqrt{2} \),

\[
\sum_{p \geq p_o \atop m \geq 2} \log p \frac{1}{p^{ms}} = \sum_{p \geq p_o} \log p \frac{1}{p^{\sigma}} \frac{1}{(1 - p^{-\sigma}) p^{2\sigma}} < c \sum_{p \geq p_o} \frac{p^{\sigma-1/2}}{p^{2\sigma}} = c \sum_{p \geq p_o} \frac{1}{p^{\sigma + 1/2}}.
\]

This suffices to prove the uniform convergence.

The dominant term \(-\sum_p \log p/p^s\) of \( \zeta'(s)/\zeta(s) \) near \( s = 1 \) now takes the form \(-D(s)\), where \( D \) is the Dirichlet-like series

\[
D(s) = \sum_{n=1}^{\infty} \frac{c_n \log n}{n^s}, \quad c_n = \begin{cases} 1 & \text{if } n \text{ is prime} \\ 0 & \text{otherwise}. \end{cases}
\]

Crucially, \( \{c_n\} \) is the same sequence that manifested in the Chebyshev theta function and the prime-counting function in section 2.2. This series is holomorphic on the open right half plane \( \text{Re}(s) > 1 \), and \( (s-1)D(s) \) extends holomorphically to an open superset of the closed right half plane \( \text{Re}(s) \geq 1 \), and \( D(s) \) extends holomorphically to this set except for a simple pole at \( s = 1 \) with residue \( \rho = 1 \). Also, the condition \( \vartheta(x) = O(x) \) is already established. These will be precisely the hypotheses for the last result of this writeup, Corollary 4.2 below, whose conclusion is then that \( \vartheta(x) \sim \rho x = x \), completing the proof of the Prime Number Theorem.

4. Convergence theorem, corollary on asymptotics, end of the proof

4.1. Theorem.

Theorem 4.1. Consider a holomorphic function \( f \) on the open right half plane \( \text{Re}(s) > 0 \), as follows: \( \alpha \) is a bounded locally integrable function on \( \mathbb{R}_{\geq 1} \), and \( f \) is the integral

\[
f(s) = \int_{t=1}^{\infty} \frac{\alpha(t)}{t^s+1} \, dt.
\]

Suppose that \( f \) extends to a holomorphic function on an open superset of the closed right half plane \( \text{Re}(s) \geq 0 \). Then the integral that defines \( f(s) \) for \( \text{Re}(s) > 0 \) converges on the closed right half plane \( \text{Re}(s) \geq 0 \).
The theorem also holds if instead \( \{a_n\} \) is a bounded sequence of complex numbers and \( f(s) \) is a Dirichlet series
\[
f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^{s+1}},
\]
by the same proof to follow.

**Proof.** It suffices to show that the integral converges at \( s = 0 \). Indeed, for any real \( y \), the function \( \tilde{f}(s) = f(s + iy) \) satisfies the same conditions as \( f \), now with \( \tilde{\alpha}(t) = \alpha(t)e^{-iyt} \) (or \( \{\tilde{a}_n\} = \{a_n/n^{iy}\} \) in the Dirichlet series case), and the convergence at \( 0 \) of the integral that initially defines \( \tilde{f} \) is precisely the convergence at \( iy \) of the integral that initially defines \( f \).

For any \( R \geq 1 \) there exists \( \delta = \delta_R > 0 \) such that \( f \) is holomorphic on the compact region determined by the conditions \( |s| \leq R \) and \( \text{Re}(s) \geq -\delta \), a truncated disk if \( \delta < R \). Consider the counterclockwise boundary of this region, consisting of an arc determined by the conditions \( |s| = R \) and \( \text{Re}(s) \geq -\delta \), and possibly a vertical segment determined by the conditions \( |s| \leq R \) and \( \text{Re}(s) = -\delta \). Typically the arc will be less than a full circle and the vertical segment will be present. Let \( A \) and \( B \) respectively denote the portions of the boundary in the right and left half planes, so that the boundary is \( A \cup B \) with \( A \) a right semicircle.

Let \( N \) be any positive integer. Because \( f(0) = f(0)^{N^0} \), Cauchy’s integral representation and Cauchy’s theorem give
\[
2\pi i f(0) = \int_{A \cup B} f(s)N^s \left( \frac{1}{s} + \frac{s}{R^2} \right) ds.
\]

Consider the \( N \)th truncation of the integral for \( f(s) \) on \( \text{Re}(s) > 0 \),
\[
f_N(s) = \int_{t=0}^{N} \alpha(t)e^{-st} dt.
\]
This is an entire function of \( s \), and so we may express its value at \( 0 \) by integrating over the circle \( A \cup -A \) rather than the truncated circle \( A \cup B \),
\[
2\pi i f_N(0) = \int_{A \cup -A} f_N(s)N^s \left( \frac{1}{s} + \frac{s}{R^2} \right) ds.
\]
Further,
\[
\int_{-A} f_N(s)N^s \left( \frac{1}{s} + \frac{s}{R^2} \right) ds = \int_{A} f_N(-s)N^{-s} \left( \frac{1}{s} + \frac{s}{R^2} \right) ds,
\]
and so in fact
\[
2\pi i f_N(0) = \int_{A} f_N(s)N^s \left( \frac{1}{s} + \frac{s}{R^2} \right) ds + \int_{A} f_N(-s)N^{-s} \left( \frac{1}{s} + \frac{s}{R^2} \right) ds.
\]
Let \( r_N = f - f_N \) denote the \( N \)th remainder, a holomorphic function on an open superset of the closed right half plane \( \text{Re}(s) \geq 0 \), represented by a tail integral on the open right half plane \( \text{Re}(s) > 0 \). Proving the theorem amounts to showing that \( \lim_N r_N(0) = 0 \). Because \( r_N = f - f_N \), the calculated expressions (4) and (5) for
$2\pi f(0)$ and $2\pi if_N(0)$ give
\[
2\pi ir_N(0) = \int_A f(s)N^s \left( \frac{1}{s} + \frac{s}{R^2} \right) \, ds + \int_B f(s)N^s \left( \frac{1}{s} + \frac{s}{R^2} \right) \, ds \\
- \int_A f_N(s)N^s \left( \frac{1}{s} + \frac{s}{R^2} \right) \, ds - \int_A f_N(-s)N^{-s} \left( \frac{1}{s} + \frac{s}{R^2} \right) \, ds,
\]
which rearranges to give $2\pi ir_N(0)$ as a sum of three terms,
\[
2\pi ir_N(0) = \int_A (r_N(s)N^s - f_N(-s)N^{-s}) \left( \frac{1}{s} + \frac{s}{R^2} \right) \, ds \\
+ \int_{B \cap \{s = R\}} f(s)N^s \left( \frac{1}{s} + \frac{s}{R^2} \right) \, ds \\
+ \int_{B \cap \{Re(s) = -\delta\}} f(s)N^s \left( \frac{1}{s} + \frac{s}{R^2} \right) \, ds.
\]

Next compute some estimates. Let $b$ be a bound of the function $\alpha$, so that $b$ depends only on $f$. Let $\sigma$ denote the real part of $s$.

- For $Re(s) > 0$, with $b$ as just above,
  \[
  |r_N(s)| \leq \frac{b}{\sigma N^\sigma}.
  \]
  Indeed, $|r_N(s)| = \left| \int_0^\infty \alpha(t)t^{-\sigma-1} \, dt \right| \leq b \int_0^\infty t^{-\sigma-1} \, dt = b/(\sigma N^\sigma)$. (In the Dirichlet series case, with $r_N(s) = \sum_{n=N} \alpha_n n^{-s-1}$, the upper bound is $b(1/N^\sigma+1/(\sigma N^\sigma))$.)

- For $Re(s) > 0$, with $b$ as above,
  \[
  |f_N(-s)| \leq \frac{b N^\sigma}{\sigma}.
  \]
  Indeed, $|f_N(-s)| = \left| \int_1^N \alpha(t)t^{-\sigma-1} \, dt \right| \leq b \int_1^N t^{-\sigma-1} \, dt = b(N^\sigma - 1)/\sigma$.

- For $|s| = R$,
  \[
  \left| \frac{1}{s} + \frac{s}{R^2} \right| = \frac{2\sigma}{R^2}.
  \]
  Indeed, $s = Re^{i\theta}$, and so $s^{-1} + sR^{-2} = (e^{i\theta} + e^{-i\theta})R^{-1} = 2R \cos \theta \cdot R^{-2}$.

- For $s$ on the vertical segment portion of $B$, because $\sigma = -\delta$ and $|s| \leq R$,
  \[
  \left| \frac{1}{s} + \frac{s}{R^2} \right| \leq \frac{1}{\delta} + \frac{1}{R} = \frac{R + \delta}{R\delta}.
  \]

From the first three estimates and from $A$ having length $\pi R$, the first term of $2\pi ir_N(0)$ in (6) satisfies
\[
\left| \int_A (r_N(s)N^s - f_N(-s)N^{-s}) \left( \frac{1}{s} + \frac{s}{R^2} \right) \, ds \right| \leq \frac{4\pi b}{R}.
\]
Let $\varepsilon > 0$ be given. If $R > 4\pi b/\varepsilon$ then $4\pi b/R < \varepsilon$.

For the given $\varepsilon > 0$, and with $R > 4\pi b/\varepsilon$ fixed, take a compatible $\delta = \delta_R > 0$, freely stipulating that $\delta < 1$, such that $f$ is holomorphic on and inside $A \cup B$. Let $M$ bound $f$ on this compact region. The second term of $2\pi ir_N(0)$ in (6) satisfies, again using the first three estimates, the conditions on $\delta$, and a little geometry,
\[
\left| \int_{B \cap \{s = R\}} f(s)N^s \left( \frac{1}{s} + \frac{s}{R^2} \right) \, ds \right| \leq \frac{8M}{R^2} \int_{\sigma = -\delta}^0 N^\sigma \, d\sigma < \frac{8M}{R^2 \log N}.
\]
If $N > \exp(8M/(R^2\varepsilon))$ then $8M/(R^2 \log N) < \varepsilon$.

Still with $\varepsilon$ and $R$ and $\delta$, the fourth estimate and the fact that $B \cap \{\text{Re}(s) = -\delta\}$ has length at most $2R$ show that the third term of $2\pi i r_N(0)$ in (6) satisfies

$$\left| \int_{B \cap \{\text{Re}(s) = -\delta\}} f(s)N^s \left( \frac{1}{s} + \frac{s}{R^2} \right) \, ds \right| \leq \frac{2M(R + \delta)}{\delta N^\delta}.$$

If $N > (2M(R + \delta)/(\delta \varepsilon))^{1/\delta}$ then $2M(R + \delta)/(\delta N^\delta) < \varepsilon$.

Altogether, given $\varepsilon > 0$, take $R > 4\pi b/\varepsilon$ and suitable $\delta = \delta_R < 1$, and then $|2\pi i r_N(0)| < 3\varepsilon$ for all large enough $N$. Thus $\{r_N(0)\}$ converges to 0, which is to say that the integral that defines $f(s)$ for $\text{Re}(s) > 0$ converges at $s = 0$ to $f(0)$. □

4.2. Corollary. The next result follows from the previous theorem.

Corollary 4.2. Let $\{c_n\}$ be a sequence of nonnegative real numbers such that the sum

$$D(s) = \sum_{n=1}^{\infty} \frac{c_n \log n}{n^s}$$

is holomorphic on the open right half plane $\text{Re}(s) > 1$. Suppose that $(s - 1)D(s)$ extends holomorphically to an open superset of the closed right half plane $\text{Re}(s) \geq 1$, so that $D(s)$ extends holomorphically to this set except possibly for a simple pole at $s = 1$. Let

$$\rho = \text{res}_1 D(s).$$

Suppose that the function

$$S(x) = \sum_{n \leq x} c_n \log n \quad \text{for real } x \geq 1$$

satisfies

$$S(x) = O(x).$$

Then

$$S(x) \sim \rho x.$$

Proof. For $\text{Re}(s) > 1$, write $D(s)$ as a Stieltjes integral, integrate by parts with the boundary terms $S(t)/t^s$ vanishing,

$$D(s) = \int_1^\infty \frac{dS(t)}{t^s} = s \int_1^\infty \frac{S(t)}{t^{s+1}} \, dt.$$

Consequently for $\text{Re}(s) > 0$, recalling the quantity $\rho = \text{res}_1 D(s)$,

$$\int_{t=1}^\infty \frac{S(t)/t - \rho}{t^{s+1}} \, dt = \frac{D(s + 1)}{s + 1} - \frac{\rho}{s}.$$

Because $D(s + 1)/(s + 1) \sim \rho/(s(s + 1)) = \rho/s - \rho/(s + 1)$ for $s$ near 0, the right side extends holomorphically to an open superset of the closed right half plane $\text{Re}(s) \geq 0$, and hence so does the left side. Further, the function $S(t)/t - \rho$ is bounded and locally integrable on $\mathbb{R}_{\geq 1}$, so it meets the conditions on the $\alpha$ in the convergence theorem. The theorem says that the integral on the left side converges for $\text{Re}(s) \geq 0$, and in particular for $s = 0$. That is,

$$\int_{t=1}^\infty \frac{S(t) - pt}{t^2} \, dt \quad \text{converges.}$$
This convergence and the fact that $S(x)$ is nonnegative and increasing show that $S(x) \sim \rho x$, meaning that $\lim_{x \to \infty} S(x)/x = \rho$, as follows. Let $\varepsilon > 0$ be given. Suppose that $S(x) \geq (1 + \varepsilon)\rho x$ for a sequence of $x$-values going to $\infty$. Estimate that for such $x$,

$$\int_{t=x}^{(1+\varepsilon)x} \frac{S(t) - \rho t}{t^2} dt \geq \int_{t=x}^{(1+\varepsilon)x} \frac{(1 + \varepsilon)\rho x - \rho t}{t^2} dt = \rho \int_{t=1}^{1+\varepsilon} \frac{1 + \varepsilon - t}{t^2} dt,$$

the last quantity positive and independent of $x$. This contradicts the convergence of the integral. Similarly, now freely taking $\varepsilon < 1$, if $S(x) \leq (1 - \varepsilon)\rho x$ for a sequence of $x$-values going to $\infty$ then for such $x$,

$$\int_{t=(1-\varepsilon)x}^{x} \frac{S(t) - \rho t}{t^2} dt \leq \rho \int_{t=1-\varepsilon}^{1} \frac{1 - \varepsilon - t}{t^2} dt,$$

negative and independent of $x$, again violating convergence. □

4.3. **End of the proof.** As noted at the end of section 3.7, the case where $c_n = 1$ if $n$ is prime and $c_n = 0$ otherwise completes the proof of the Prime Number Theorem. In this case, $D(s)$ is (minus) the dominant term of the logarithmic derivative $\zeta'(s)/\zeta(s)$, with residue $\rho = 1$ at $s = 1$, and $S(x)$ is the Chebyshev theta function $\vartheta(x)$, known to be $O(x)$. The asymptotic result $\vartheta(x) \sim x$ from Corollary 4.2 is exactly what is needed to finish the Prime Number Theorem argument.