MATH 311: COMPLEX ANALYSIS — CONFORMAL MAPPINGS LECTURE

1. Introduction

Let D denote the unit disk and let ∂D denote its boundary circle. Consider a piecewise continuous function on the boundary circle,

$$\varphi : \partial D \longrightarrow \mathbb{R}, \qquad \varphi(z) = \begin{cases} 0 & \text{if } \mathrm{Re}(z) \geq 0, \\ 1 & \text{if } \mathrm{Re}(z) < 0. \end{cases}$$

A special case of *Dirichlet's Problem* is as follows: *Find a harmonic function*

$$h : D \longrightarrow \mathbb{R}$$

that extends continuously to φ on ∂D except for the points where φ itself is discontinuous.

Here is a solution to the problem. We assert without proof for now that the function

$$f(z) = \frac{z+i}{iz+1}$$

takes the disk D to the upper half plane $\mathcal{H} = \{z : \mathrm{Im}(z) > 0\}$, and furthermore f takes the right half of the boundary circle ∂D to the positive real axis and f takes the left half of ∂D to the negative real axis. We soon will study a class of functions having the same form as f, so for now we may take the properties of f for granted.

Next consider a branch of the complex logarithm function defined off zero and the negative imaginary axis and taking the argument on the region that remains to lie in $(-\pi/2, 3\pi/2)$,

$$\log : \mathbb{C} - \{iy : y \leq 0\} \longrightarrow \{z \in \mathbb{C} : -\pi/2 < \arg(z) < 3\pi/2\}.$$

This branch of log includes in its domain the upper half plane and the punctured real axis. Its imaginary part is

$$\tilde{h} : \mathbb{C} - \{iy : y \leq 0\} \longrightarrow (-\pi/2, 3\pi/2).$$

As the imaginary part of an analytic function, $\tilde{h}$ is harmonic. And $\tilde{h} = 0$ on the positive real axis, while $\tilde{h} = \pi$ on the negative real axis. Thus the composition

$$h = \frac{1}{\pi}\,\tilde{h} \circ f : D \cup \partial D - \{\pm i\} \longrightarrow \mathbb{R}$$

is the imaginary part of the analytic function $(1/\pi)\log \circ f$ on D, and it takes the desired values 0 on the left half of ∂D and 1 on the right half. In sum, h solves our particular case of Dirichlet's Problem.

The key ingredient in the solution was the function f that took the disk to the upper half plane, also taking the two halves of the boundary circle to two convenient segments of the real axis and thus making the problem easy to solve. This f is an example of a *conformal mapping*.

2. Definition and Characterization

Definition 2.1. *Let $\Omega \subset \mathbb{R}^2$ be a region, and let*

$$f : \Omega \longrightarrow \mathbb{R}^2$$

be a $\mathcal{C}^1$-mapping. That is, $f(x,y) = u(x,y) + iv(x,y)$ where $u, v : \Omega \longrightarrow \mathbb{R}$ have continuous partial derivatives.

Let (x_0, y_0) be any point of Ω. Then f is **conformal at** (x_0, y_0) *if there exist numbers*

$$r \in \mathbb{R}^+, \qquad \theta \in [0, 2\pi)$$

such that the following condition holds: For any differentiable curve in the region and passing through the point,

$$\gamma : I \longrightarrow \Omega, \qquad \gamma(0) = (x_0, y_0),$$

we have

$$|(f \circ \gamma)'(0)| = r \cdot |\gamma'(0)|, \qquad \arg((f \circ \gamma)'(0)) = \theta + \arg(\gamma'(0)).$$

Finally, f is **conformal** *if it is conformal at each point of Ω.*

That is, when any differentiable curve through (x_0, y_0) is passed through f, its tangent vector at (x_0, y_0) is stretched by a factor r and rotated through an angle θ, where r and θ are independent of the curve. A conformal map scales and rotates all tangent vectors at a point uniformly, independently of their lengths or directions..

As always, we may identify $\mathbb{R}^2$ with $\mathbb{C}$ and view the previous definition as applying to complex-valued functions on complex domains. The next proposition shows the advantage of doing so: in complex analytic terms, conformality is nothing new — it is simply differentiability.

Proposition 2.2. *Let $\Omega \subset \mathbb{C}$ be a region, and let*

$$f : \Omega \longrightarrow \mathbb{C}$$

be a $\mathcal{C}^1$ as a mapping to $\mathbb{R}^2$. Let z_0 be any point of Ω. Then f is conformal at z_0 if and only if f is complex-differentiable at z_0 and $f'(z_0) \neq 0$.

Proof. Because f is $\mathcal{C}^1$ on Ω, it is vector-differentiable on Ω. Its derivative matrix $f'(z_o)$ at z_o is two-by-two with real entries. For any curve γ as in the previous definition,

$$(f \circ \gamma)'(0) = f'(z_o)\gamma'(0).$$

The condition for $f'(z_o)$ to be a positive dilation of a rotation of $\gamma'(0)$ for all nonzero $\gamma'(0)$ is the condition

$$f'(z_o) = r \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{for some } r \in \mathbb{R}^+ \text{ and } \theta \in [0, 2\pi).$$

This is the condition

$$f'(z_o) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \quad \text{for some } a, b \in \mathbb{R}, \text{ not both zero,}$$

which is the condition that f be complex-differentiable at z_o with $f'(z_o) \neq 0$. □

For example, the exponential function is conformal on all of $\mathbb{C}$. Every nth power function is conformal except at 0. Every branch of the complex logarithm or of any nth root function is conformal on its domain (which cannot include branch points). A separate writeup shows that stereographic projection is conformal, although the writeup is using a more general idea of conformality than we have in play here.

It may worth distinguishing explicitly between the *round sphere*

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$$

and the Riemann sphere

$$\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}.$$

Topologically, the Riemann sphere is the *one-point compactification* of the complex plane. Analytically, it is a *compact Riemann surface*, meaning a connected one-dimensional complex manifold. The local coordinate function at a point $c \in \widehat{\mathbb{C}}$ is

$$\begin{cases} \varphi_c : \widehat{\mathbb{C}} - \{\infty\} \longrightarrow \mathbb{C}, & \varphi_c(z) = z - c \quad \text{if } c \in \mathbb{C}, \\ \varphi_\infty : \widehat{\mathbb{C}} - \{0\} \longrightarrow \mathbb{C}, & \varphi_\infty(z) = 1/z \quad \text{if } c = \infty. \end{cases}$$

(Here $1/\infty$ is understood to mean 0.) Geometrically, the Riemann sphere is the complex plane augmented with one more point conceptually infinitely far away in all directions. Algebraically, it is the *complex projective line*, to be mentioned again soon.

3. Fractional Linear Transformations

Consider any invertible 2-by-2 matrix with complex entries,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad a, b, c, d \in \mathbb{C}, \ ad - bc \neq 0.$$

This matrix acts on the Riemann sphere as follows:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}(z) = \begin{cases} \dfrac{az+b}{cz+d} & \text{if } z \neq \infty \text{ and } cz + d \neq 0, \\ \dfrac{a}{c} & \text{if } z = \infty \text{ and } c \neq 0, \\ \infty & \text{if } z \neq \infty \text{ and } cz + d = 0, \\ \infty & \text{if } z = \infty \text{ and } c = 0, \end{cases}$$

or more succinctly as the generic case, with the other cases being tacit,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}(z) = \frac{az+b}{cz+d}.$$

The action is associative, meaning that for all suitable pairs of matrices and all points z of the Riemann sphere,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\left(\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}(z)\right) = \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}\right)(z).$$

As matters stand, verifying this associativity requires checking eight cases. Most people just check the case not involving the point ∞ anywhere and then take the other cases for granted. A separate writeup shows how to think about the problem in more conceptual terms. One key idea in the writeup is that in algebraic contexts, we emphatically should think of the Riemann sphere as the complex projective line.

4. Fractional Linear Transformations are Conformal

Again consider any invertible 2-by-2 matrix with complex entries,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad a, b, c, d \in \mathbb{C}, \ ad - bc \neq 0.$$

Let z_0 be a point of the Riemann sphere. If $z_0 \neq \infty$ and $cz_0 + d \neq 0$ then compute the derivative

$$\left(\frac{az+b}{cz+d}\right)'\Big|_{z=z_0} = \frac{a(cz+d)-(az+b)c}{(cz+d)^2}\Big|_{z=z_0} = \frac{ad-bc}{(cz_0+d)^2} \neq 0.$$

Thus the mapping defined by the matrix is conformal at z_0. If $z_0 = \infty$ and $c \neq 0$ then the calculation needs to work instead with inputs in the appropriate local coordinate system,

$$\left(\frac{az+b}{cz+d}\right)'\Big|_{z=\infty} = \left(\frac{a/\zeta+b}{c/\zeta+d}\right)'\Big|_{\zeta=0} = \left(\frac{b\zeta+a}{d\zeta+c}\right)'\Big|_{\zeta=0} = \frac{bc-ad}{c^2} \neq 0.$$

On the other hand, if $z_0 \neq \infty$ and $cz_0 + d = 0$ then the calculation needs to work with outputs in the appropriate local coordinate system,

$$\left(\frac{cz+d}{az+b}\right)'\Big|_{z=z_0} = \frac{bc-ad}{(az_0+b)^2} \neq 0.$$

Here we know that $az_0+b \neq 0$ because already $cz_0+d = 0$, so that if also $az_0+b = 0$ then the matrix equation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} z_0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

makes the assumed condition $ad - bc \neq 0$ impossible. Finally, if $z_0 = \infty$ and $c = 0$ then the calculation needs to work with inputs and outputs both in the appropriate local coordinate system,

$$\left(\frac{az+b}{d}\right)'\Big|_{z=\infty} = \left(\frac{d}{a/\zeta+b}\right)'\Big|_{\zeta=0} = \left(\frac{d\zeta}{b\zeta+a}\right)'\Big|_{\zeta=0} = \frac{da}{a^2} = \frac{d}{a} \neq 0.$$

Here we know that a and d are nonzero by the assumed conditions $ad - bc \neq 0$ and $c = 0$. In sum, the fractional linear transformation defined by the matrix is conformal everywhere on the Riemann sphere, including the points of infinite input or output.

5. A Fact About Fractional Linear Transformations

Fractional linear transformations take circles to circles, where the notion of circles in the Riemann sphere encompasses lines, viewed as circles through ∞. This fact is shown conceptually in a separate writeup.

Here is a more computational demonstration of the same fact. Consider any 2-by-2 complex matrix with nonzero determinant, viewed as a fractional linear transformation,

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad a, b, c, d \in \mathbb{C}, \ ad - bc \neq 0.$$

If $c = 0$ then g is an affine transformation, $g : z \mapsto (a/d)z + (b/d)$. Otherwise, let $\Delta = ad - bc$ and compute that

$$g = pwn, \qquad p = \begin{bmatrix} \Delta/c & a \\ 0 & c \end{bmatrix}, \ w = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \ n = \begin{bmatrix} 1 & d/c \\ 0 & 1 \end{bmatrix},$$

showing that g is the composition of a translation, the negative-reciprocal map, and an affine transformation. It follows from this discussion that if affine fractional linear transformations and the negative-reciprocal fractional linear transformation w take circles to circles, then all fractional linear transformations do so.

The equation of a circle centered at C and having radius R is

$$|z|^2 - 2\mathrm{Re}(\overline{C}z) + |C|^2 - R^2 = 0,$$

or

$$|z|^2 + \bar{b}z + b\overline{z} + c = 0, \quad b \in \mathbb{C}, \ c \in \mathbb{R}.$$

And by a homework problem, the equation of a line is

$$\bar{b}z + b\overline{z} + c = 0, \quad b \in \mathbb{C}, \ c \in \mathbb{R}, \ b \neq 0.$$

With these equations in hand, it is not hard to verify that affine fractional linear transformations take circles to circles and lines to lines, with the negative-reciprocal fractional linear transformation takes circles through 0 to lines, lines to circles through 0, and otherwise takes circles to circles.

6. Triple Transitivity

Any nonidentity fractional linear transformation fixes at most two points. Indeed, the condition for a fixed point z is $az + b = z(cz + d)$, or

$$cz^2 - (a - d)z - b = 0.$$

This condition is quadratic if $c \neq 0$ and linear if $c = 0$ and $a \neq d$, (but note that in the linear case, $z = \infty$ is a fixed point as well). If $c = 0$ and $a = d$ then the condition is $b = 0$, giving no solutions z if $b \neq 0$ and giving every z-value as a solution if $b = 0$. (From the correct perspective, the real issue here is that the linear transformation has at most two eigenspaces.)

Let z_1, z_2, z_3 be any three distinct points of the Riemann sphere. Consider the fractional linear transformation

$$Tz = \begin{cases} \dfrac{z - z_2}{z - z_3} \cdot \dfrac{z_1 - z_3}{z_1 - z_2} & \text{if } z_1, z_2, z_3 \text{ are finite,} \\ \dfrac{z - z_2}{z - z_3} & \text{if } z_1 = \infty, \\ \dfrac{z_1 - z_3}{z - z_3} & \text{if } z_2 = \infty, \\ \dfrac{z - z_2}{z_1 - z_2} & \text{if } z_3 = \infty. \end{cases}$$

Then in all cases,

$$Tz_1 = 1, \quad Tz_2 = 0, \quad Tz_3 = \infty.$$

If also w_1, w_2, w_3 are three distinct points of the Riemann sphere then a similarly-defined fractional linear transformation U satisfies

$$Uw_1 = 1, \quad Uw_2 = 0, \quad Uw_3 = \infty.$$

And therefore, the composition $V = U^{-1}T$ satisfies

$$Vz_1 = w_1, \quad Vz_2 = w_2, \quad Vz_3 = w_3.$$

Thus we can take any three points to any three points. And the fractional linear transformation V that does so is unique, because if also another one $\widetilde{V}$ does so as well then the composition $V^{-1} \circ \widetilde{V}$ fixes three points, making it the identity.

Returning to the transformation T above, the general value Tz is the *cross-ratio* of the four points z, z_1, z_2, and z_3, denoted (z, z_1, z_2, z_3). A homework problem will show that the cross-ratio is preserved by fractional linear transformations, and so we can use a fractional linear transformation to take four points to four points exactly when the two quadruples have the same cross ratio. Also, one can use the cross ratio to show computationally that fractional linear transformations take circles to circles.

The formula for *reflection in a circle* is

$$\widehat{z} - c = \frac{r^2}{|z-c|^2}(z-c) = \frac{r^2}{\overline{z} - \overline{c}}.$$

7. Examples of Conformal Maps

- Find a conformal equivalence $\mathcal{H} \stackrel{\sim}{\longrightarrow} D$. The fractional linear transformation

$$z \longmapsto (z, 1, i, -i) = \frac{z-i}{z+i} \cdot \frac{1+i}{1-i} = i \cdot \frac{z-i}{z+i} = \frac{z-i}{-iz+1} = \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}(z)$$

  will do. It suggests itself as the rotation of the round sphere about the positive x-axis clockwise by $\pi/2$; it is the inverse of the map from the disk to the upper half plane that was used in the initial example of this writeup. Normalizing to determinant 1, the matrix for the initial example is

$$m = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \quad \text{(the \textit{Cayley element})},$$

  a unitary matrix (i.e., $m^* m = I_2$). The connection between rotations of the round sphere and unitary matrices will be explained in another writeup.
- Let $\alpha < \beta$ be real numbers, and let

$$A = \{z \in \mathbb{C} : \alpha < \arg(z) < \beta\}.$$

  (Since *arg* is multiple-valued, one applies a smidgeon of common sense to interpret the previous display. Either choose a suitable branch of *arg*, or interpret the condition to mean that *some* value of $\arg(z)$ lies in the specified range.) To find a conformal equivalence

$$A \stackrel{\sim}{\longrightarrow} D,$$

  note that the map

$$z \longmapsto (e^{-i\alpha} z)^{\pi/(\beta-\alpha)}.$$

  is a conformal equivalence $A \stackrel{\sim}{\longrightarrow} \mathcal{H}$, and so we are done by the previous example. The multiply-by-constant map rotates the sector so that the argument-range becomes $(0, \beta - \alpha)$, and then the power map opens the angle to π. The multiply-by-constant map is conformal because it is differentiable with nonzero derivative everywhere, and the power map is conformal because it is differentiable with nonzero derivative everywhere except the origin. On the round sphere, a lune is being spun longitudinally to begin at zero, and then it is being opened up to the right half-sphere.

- Let C and C' be circles in the plane with two points of intersection z_1 and z_2, and let Ω be the intersection of their interiors. To find a conformal equivalence
$$\Omega \xrightarrow{\sim} D,$$
note that the fractional linear transformation
$$z \longmapsto \frac{z - z_1}{z - z_2}$$
takes Ω to a sector, and so we are done the previous example. (Admittedly, one has to do a little more work here to specify the sector precisely, etc.) On the round sphere, the region between two circles is being moved to a lune.
- Let S be any strip in the complex plane. A conformal equivalence
$$S \xrightarrow{\sim} D$$
is given by the composition of the map
$$z \longmapsto \exp(az + b),$$
followed by the first example of this section, where the affine map $z \mapsto az+b$ moves the general strip to the specific strip $S' = \{z \in \mathbb{C} : 0 < \mathrm{Im}(z) < \pi\}$. The exponential map is differentiable with nonzero derivative, making it conformal, and any branch of *log* whose argument-range includes $(0, \pi)$ will serve as an inverse. On the round sphere, the strip is the region between two circles that are parallel at the north pole. The specific strip involves a longitudinal circle and a circle in the right half-sphere. The exponential map takes two orthogonal families of circles through the north pole (those parallel to longitude 0 and those parallel to longitude $\pi/2$ to longitudinal and latitudinal circles.
- Let C and C' be circles in the plane with one point of intersection. Let B be the pincers-shaped region interior to the larger circle and exterior to the smaller one. To find a conformal equivalence
$$B \xrightarrow{\sim} D,$$
note that the line containing the centers of both circles thus intersects the smaller circle at points a and b, the larger circle at the same a and at a third point c. The fractional linear transformation
$$z \longmapsto (z, c, b, a)$$
takes B to the vertical strip $\{z \in \mathbb{C} : 0 < \mathrm{Re}(z) < 1\}$, and so we are done by the previous example. On the round sphere, the pincers are being moved to pince the north pole.
- Let $R = \mathcal{H} \setminus \overline{D}$. To fined a conformal equivalence
$$R \longrightarrow \mathcal{H},$$
the trick is *not* to think that since R is nearly all of $\mathcal{H}$, all we need to do is tug a little. The idea, as always, is to think in terms of circles and angles. Thus the fractional linear transformation
$$z \longmapsto (z, i, -1, 1)$$
takes R to the first quadrant, and then the squaring map finishes the job. On the round sphere, the upper right quarter-sphere is being rotated to the

front right quarter-sphere, about the ray from the origin through $(1,1,1)$, and then the squaring map opens this up to the right half-sphere. One can see the rotation of the round sphere by noting that the fractional linear transformation is

$$z \longmapsto \frac{z+1}{z-1} \cdot \frac{i-1}{i+1} = i\,\frac{z+1}{z-1}\,,$$

which takes ∞ to i, and so it permutes i, 1, and ∞ cyclically.

- To find a conformal equivalence

$$\mathbb{C} \setminus (\overline{D} \cup \mathbb{R}^+) \longrightarrow \mathcal{H},$$

note that the square root (where $0 < \arg < 2\pi$) reduces this immediately to the previous problem.

- Find a fractional linear transformation taking the circle $\{|z| = 4\}$ to the circle $\{|z - i| = 1\}$, taking $4 \mapsto 0$, and taking $0 \mapsto 2$.

Let $a = 4$ and let $b = 0$. We need only determine where one more point c is taken. A natural point to consider is $c = -4$, since this point lies on the circle $\{|z| = 4\}$ and on the line through a and b. By geometry, $c \mapsto 1 + i$. So if we define the transformations

$$Tz = (z, 4, 0, -4), \qquad Uw = (w, 0, 2, 1 + i)$$

then the desired map is $U^{-1}T$. A short (and not entirely pleasant) calculation with 2-by-2 matrices shows that in fact the map is

$$z \longmapsto \frac{-2z+8}{(-1+2i)z+4}\,.$$

Another way to solve this problem is by using properties of reflection in circles. Specifically, if T is a fractional linear transformation, and γ is a circle, and z is a point, then then the image-point of the reflection through the circle is the reflection through the image circle of the image-point,

$$T(\hat{z}_\gamma) = \widehat{(Tz)}_{T\gamma}.$$

Returning to the problem, now let $\gamma = \{|z| = 4\}$ and let $c = \infty = \hat{0}_\gamma$, so that $T\infty$ must be the reflection of $T0 = 2$ through $T\gamma = \{|z - i| = 1\}$. The reflection formula gives

$$\hat{2}_{T\gamma} - i = \frac{1}{5}(2 - i),$$

and so in fact $T\infty = (2 + 4i)/5$. Now another matrix calculation finds the desired fractional linear transformation, and it inevitably is the same as before.

- Find a fractional linear transformation taking the two nonintersecting circles $\{|z - c_j| = r_j\}$ $(j = 1, 2)$ to concentric circles about the origin of radii 1 and r where $r > 1$. What is r?

Let the line ℓ through the centers of the two circles meet the first circles at the points a and b, the second circle at c and d. Label the points so that traversing the line from a to b and then onward encounters c before d. We will map ℓ to the real axis, taking

$$a \longmapsto 1, \quad b \longmapsto -1, \quad c \longmapsto -r, \quad d \longmapsto r.$$

(A slightly subtle point here is that the *only* way to solve the problem is to map ℓ to the real axis.) By a homework problem, there exists a fractional linear transformation taking one 4-tuple to another exactly when the cross ratios match,

$$(a, b, c, d) = (1, -1, -r, r).$$

That is, the condition is

$$\frac{a-c}{a-d} \cdot \frac{b-d}{b-c} = \frac{1+r}{1-r} \cdot \frac{-1-r}{-1+r} = \frac{(1+r)^2}{(1-r)^2}\,.$$

Thus the condition is quadratic in r, taking the form

$$C(1-r)^2 = (1+r)^2, \qquad C \in \mathbb{R} - \{0, 1\}.$$

(The cross-ratio C can't be 1 because it is Ta/Tb where $Tz = \frac{z-c}{z-d}$, and a and b are distinct.) One of the roots will be greater than 1, and it is the solution.

On the round sphere, the problem is to take the region between two circles to the region between the equator and a line of constant latitude in the upper hemisphere. The latitude is uniquely determined.

8. Dynamics of Fractional Linear Transformations

View fractional linear transformations as 2-by-2 complex matrices with determinant 1. Given such a matrix m, there is an invertible matrix p such that

$$p^{-1}mp = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \qquad \text{or} \qquad p^{-1}mp = \begin{bmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{bmatrix}.$$

Assume that the fractional linear transformation is *not* the identity mapping.

- In the first case of the display, if λ is real, the transformation is called *hyperbolic*. It acts as
$$z \longmapsto \lambda^2 z, \quad \lambda^2 = r \in \mathbb{R}^+,$$
giving a dilation. (If $\lambda^2 < 1$, the dilation is really a contraction, but in any case we can exchange λ and λ^{-1} by exchanging the columns of the conjugating matrix p.) The transformation moves points outward or inward along rays from the origin, permuting concentric circles around the origin.
- In the first case of the display, if $|\lambda| = 1$, the transformation is called *elliptic*. It acts as
$$z \longmapsto \lambda^2 z, \quad \lambda^2 = e^{i\theta} \in \{\text{unit circle}\},$$
giving a rotation. The transformation moves points along concentric circles around the origin, permuting rays from the origin.
- In the second case of the display, the transformation is called *parabolic*. It acts as
$$z \longmapsto z \pm 1,$$
giving a translation. The transformation moves points along parallel horizontal lines, permuting parallel vertical lines.
- Otherwise the transformation is called *loxodromic*.

Undoing the change of coordinates and iterating the transformation many times, we see that

- A hyperbolic transformation has two fixed points, one attracting and one repelling. The paths of motion are a family of circles through the two points. Each iteration of the transformation permutes a second family of circles orthogonal to the first family.
- An elliptic transformation has two fixed points, neither of them attracting or repelling. Each iteration of the transformation permutes a family of circles through the two points. The paths of motion are a second family of circles orthogonal to the first family.
- A parabolic transformation has one fixed point, attracting in one direction and repelling in the opposite direction. The paths of motion are a family of circles through the point. Each iteration of the transformation permutes a second family of circles through the point, orthogonal to the first.