MATHEMATICS 311: COMPLEX ANALYSIS — ASSIGNMENT 8

Reading: Marsden, sections 5.1, 5.2.

Problems:

1. Consider a nonidentity fractional linear transformation

$$Sz = \begin{bmatrix} a & b \\ c & d \end{bmatrix}(z), \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PSL}_2(\mathbb{C}).$$

Show that S can be classified as elliptic, parabolic, hyperbolic, or loxodromic solely in terms of the projective trace $\pm(a+d)$, which is well defined. For example, S is elliptic if and only if $a+d$ is real and $|a+d|<2$.

2. The *cross-ratio* of any four distinct points $z_1, z_2, z_3, z_4 \in \mathbb{C}$ is defined as

$$(z_1, z_2, z_3, z_4) = \frac{z_1 - z_3}{z_1 - z_4} \cdot \frac{z_2 - z_4}{z_2 - z_3}.$$

(a) Show that $(z_1, z_2, z_3, z_4) = Sz_1$, where S is the unique fractional linear transformation such that $Sz_2 = 1$, $Sz_3 = 0$, $Sz_4 = \infty$.

(b) Use part (a) to show that $(z_1, z_2, z_3, z_4) = (Tz_1, Tz_2, Tz_3, Tz_4)$ for any fractional linear transformation T.

(c) Use part (b) to show that (z_1, z_2, z_3, z_4) is real precisely when these four points lie either on the same line or the same circle.

3. (a) Show that the reflection z^* of the point z in the circle passing through three distinct points z_1, z_2, z_3 is characterized by the condition

$$(z^*, z_1, z_2, z_3) = \overline{(z, z_1, z_2, z_3)}.$$

(Hint for one direction: If the circle through z_1, z_2, z_3 is centered at a and has radius R then using properties of the cross-ratio shows that

$$\begin{aligned} \overline{(z, z_1, z_2, z_3)} &= \overline{(z-a, z_1-a, z_2-a, z_3-a)} \\ &= (\overline{z}-\overline{a}, R^2/(z_1-a), R^2/(z_2-a), R^2/(z_3-a)) \\ &= (R^2/(\overline{z}-\overline{a}), z_1-a, z_2-a, z_3-a) \\ &= (R^2/(\overline{z}-\overline{a})+a, z_1, z_2, z_3). \end{aligned}$$

Since I am giving you the calculation, explain the steps.)

(b) Suppose that z^* is the reflection of z in a circle C. Show that Sz^* is the reflection of Sz in the circle SC.

4. Find fractional linear transformations:

(a) R, taking the half-disk $\{z \in \mathbb{C} : |z| < 1, \mathrm{Im}(z) > 0\}$ to the first quadrant $\{z \in \mathbb{C} : \mathrm{Re}(z) > 0, \mathrm{Im}(z) > 0\}$.

(b) S, taking the circle $|z| = 2$ to the circle $|z+1| = 1$, the point -2 to the origin, and the origin to i.

(c) T, taking the circles $|z| = 1$ and $|z - 1/4| = 1/4$ to $|z| = R$ and $|z| = 1$ respectively, with $R > 1$, and taking $1/2 \mapsto 1$. What must R be?