MATHEMATICS 311: COMPLEX ANALYSIS — ASSIGNMENT 7

Reading: Marsden, 4.1, 4.2, 4.3, 6.2, 6.3.
Problems:

1. Evaluate the following integrals by using residue techniques (in each case you have a template available from the lecture):

(a) $\int_{-\infty}^{\infty} \frac{x^2\,\mathrm{d}x}{x^4+5x^2+6}$,

(b) $\int_0^{\infty} \frac{x^2\,\mathrm{d}x}{(x^2+a^2)^3}$ where $0 < a \in \mathbb{R}$,

(c) $\int_0^{\infty} \frac{\cos ax\,\mathrm{d}x}{1+x^2}$ where $0 < a \in \mathbb{R}$,

(d) $\int_0^{2\pi} \frac{\mathrm{d}\theta}{a+b\sin\theta}$ where $a, b \in \mathbb{R}$ and $a > |b| > 0$,

(e) $\int_0^{\infty} \frac{\sqrt[3]{x}\,\mathrm{d}x}{1+x^2}$.

2. Let $f(z)$ be analytic inside and on an ellipse γ with foci at $\{-1, 1\}$. Show that

$$\frac{z+1}{z-1} \in [-\infty, 0] \iff z \in [-1, 1].$$

Consequently, one can define $\log\left(\frac{z+1}{z-1}\right)$ as a single-valued function on γ, taking

$$\arg\left(\frac{z+1}{z-1}\right) = \arg(z+1) - \arg(z-1),$$

even though neither argument is well defined by itself. Show that

$$\int_\gamma f(z)\log\left(\frac{z+1}{z-1}\right)\mathrm{d}z = 2\pi i \int_{x=-1}^{1} f(x)\,\mathrm{d}x$$

where the right hand expression is a real integral, i.e., x runs from -1 to 1 in $\mathbb{R}$. (Begin by deforming the path of integration to a barbell. What is the argument of $(z+1)/(z-1)$ on the top of the bar, and on the bottom? How does the integrand grow as the two end-circles shrink, and how does that compare to the length of the end-circles? The Residue Theorem plays no role in this problem.)

3. How many roots of the equation $z^4 - 6z + 3 = 0$ have their modulus between 1 and 2?

4. If $a \in \mathbb{R}$ and $a > e$, show that the equation $e^z = az^n$ has n solutions in the disk $\{z : |z| < 1\}$.