# MATHEMATICS 311: COMPLEX ANALYSIS — ASSIGNMENT 2

**Reading**: Marsden, sections 1.3, 1.5, 1.6.

**Problems**:

1. Show that if $f(z)$ is analytic in a region $\Omega$ and either $\mathrm{Re} f(z)$ or $|f(z)|$ is constant in $\Omega$ then $f(z)$ must be constant there.

2. Show that if $f(z)$ is analytic and its second partial derivatives exist and are continuous then $\Delta(|f(z)|^2) = 4|f'(z)|^2$, where $\Delta$ is the Laplacian operator, $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$.

3. Show that the function

$$f(z) = \begin{cases} e^{-1/z^4} & \text{for } z \neq 0 \\ 0 & \text{for } z = 0 \end{cases}$$

is analytic at all $z \neq 0$, is not analytic at $z = 0$, but satisfies the Cauchy–Riemann equations at $z = 0$. (For $z \neq 0$, decompose $f$ as a composition of functions each known to be analytic; to show $f$ is not analytic at 0 it suffices to show $f$ is not even continuous at 0; the form $f_x = -if_y$ of the Cauchy–Riemann equations is easiest to check at 0.)

4. Extend the trigonometric functions to complex arguments by defining

$$\sin z = \sin x \cosh y + i \cos x \sinh y$$
$$\cos z = \cos x \cosh y - i \sin x \sinh y$$

for all $z = x + iy \in \mathbb{C}$; here the hyperbolic functions are defined as usual by $\sinh y = (e^y - e^{-y})/2$, $\cosh y = (e^y + e^{-y})/2$ for $y \in \mathbb{R}$.

(a) Show that $\sin z$ is analytic for all $z \in \mathbb{C}$ and find all points $z$ for which $\sin z = 0$.

(b) Show: $\cos(\pi/2 - z) = \sin z$, $\cos(z + \pi) = -\cos z$, $\cos(z + 2\pi) = \cos z$, $\sin^2 z + \cos^2 z = 1$, $\cos z = (e^{iz} + e^{-iz})/2$ for all $z \in \mathbb{C}$.

(c) Discuss the mapping described by the function $w = \cos z$. It suffices to consider the strip $0 \leq \mathrm{Re} z \leq 2\pi$, by periodicity. The cosine function can be written as the composition of the functions $z \to iz$, $z \to e^z$, $z \to (z + z^{-1})/2$, each of which is familiar. Illustrate suitable restrictions of each of these functions to give a good sense of the composite. On what part of the strip is cos 1-to-1? What is its output?

(d) Discuss a single-valued inverse cosine function: where it is defined, a formula for it in terms of the logarithm function, what its branch points are.