# THEORY OF FUNCTIONS

## 1. Introduction

## 2. Local Analysis of Analytic Functions

**Theorem 2.1** (Local Mapping Theorem). *Suppose $f$ is analytic at $z_0$ and that $f(z) - w_0$ has a zero of order $n$ at $z_0$. Then for all sufficiently small $\varepsilon > 0$ there exists $\delta > 0$ such that for all $w \in N(w_0;\delta) \setminus \{w_0\}$, the equation $f(z) = w$ has $n$ distinct roots in $N(z_0;\varepsilon)$. In other words, $f$ is $n$–to–1 near $z_0$.*

*Proof.* Since $f$ is not identically $w_0$, the $w_0$–points of $f$ are isolated, hence in some $N(z_0; 2\varepsilon)$, $f$ takes the value $w_0$ only at $z_0$. Let $\hat{\gamma} = \{z : |z - z_0| = \varepsilon\}$, traversed once counterclockwise, and let $\Gamma = f \circ \gamma$. Then $n = 1/2\pi i \int_\gamma f'(\zeta)d\zeta/(f(\zeta) - w_0) = 1/2\pi i \int_\Gamma d\xi/(\xi - w_0) = F(w_0)$, where $F(w) = 1/2\pi i \int_\Gamma d\xi/(\xi - w)$. Since $f \neq w_0$ on $\hat{\gamma}$, $w_0 \notin \hat{\Gamma}$, so some $N(w_0; 2\delta)$ does not intersect $\hat{\Gamma}$, hence some $N(w_0, \delta)$ has all points at least distance $\delta$ from $\hat{\Gamma}$.

$F$ is continuous at $w_0$. Proof: By the standard estimate and a little algebra, for $w \in N(w_0;\delta)$,

$$|F(w) - F(w_0)| \leq \frac{1}{2\pi} \int_\Gamma \frac{|w - w_0||d\xi|}{(\xi - w)(\xi - w_0)} \leq \frac{|w - w_0|\text{length}(\Gamma)}{\delta^2},$$

so if also $w \in N(w_0; 2\pi\delta^2/\text{length}(\Gamma))$ then $|F(w) - F(w_0)| < \varepsilon$.

Since $F(w)$ is the number of $w$–points of $f$ inside $\hat{\gamma}$, $F$ is integer–valued and (by the claim) continuous at $w_0$. By uniqueness, $F$ must be the constant function $F(w) = n$ on $N(w_0;\delta)$, i.e., $f(z) = w$ has $n$ solutions for $w \in N(w_0;\delta)$. These solutions are distinct because since the zeros of $f'$ are isolated, we may take $\varepsilon$ also small enough that $f'(z) \neq 0$ in $N(z_0;\varepsilon) \setminus \{z_0\}$. □

A corollary is

**Theorem 2.2** (Open Mapping Theorem). *Suppose $f$ is analytic, nonconstant. Then $f$ maps open sets to open sets, and at $z_0$ such that $f'(z_0) \neq 0$, $f$ is a local homeomorphism.*

*Proof.* Open mapping: Say $S$ is some open set in the domain of $f$ and $w_0 \in f(S)$, i.e., $w_0 = f(z_0)$ for some $z_0 \in S$. Then for all small $\varepsilon$, $N(z_0;\varepsilon) \subset S$, and for sufficiently small $\varepsilon$, some $N(w_0;\delta) \subset f(N(z_0;\varepsilon) \subset f(S)$. This shows that $f(S)$ is open.

Local homeomorphism: If $n = 1$ in the local mapping theorem, $f$ gives a bijection between $N(w_0;\delta)$ and $f^{-1}(N(w_0;\delta))$. $f^{-1}$ is continuous by the open set property of $f$. □

(In fact, $f^{-1}$ is also analytic. See Knopp, p.136.)

Note how this reproves the maximum principle more convincingly: analytic $f$ maps blobs to blobs, so $|f|$ can't take a maximum on a blob.

To make all this explicit: Near $z_0$, $f(z) - w_0 = (z - z_0)^n g(z)$, where $g(z_0) \neq 0$, hence $g(z) \neq 0$ for $z$ near $z_0$ by continuity. So we can take an $n^{th}$ root of $g(z)$

near $z_0$, call it $h(z)$. Thus $w - w_0 = f(z) - w_0 = ((z-z_0)h(z))^n = \zeta^n$, where $\zeta = k(z) = (z-z_0)h(z)$, a homeomorphism near $z_0$. Thus the general map $f$ is locally a translation of a homeomorphism followed by an $n^{th}$ power.

## 3. Behavior at infinity

If $f$ is analytic on $\mathbf{C}$ except for finitely many singularities then $f$ is analytic on $\{|z| > \rho\}$ for some positive $\rho$. Define

$$g : \{\zeta : 0 < |\zeta| < 1/\rho\} \longrightarrow \mathbf{C}, \qquad g(\zeta) = f(1/\zeta).$$

Then $g$ has an isolated singularity at 0. The nature of the singularity of $f$ at $\infty$ is *defined* to be the nature of the singularity of $g$ at 0. For example, a polynomial of degree $n$ has a singularity of degree $n$ at $\infty$ since in this case

$$g(\zeta) = p(1/\zeta) = \sum_{k=0}^{n} a_k \zeta^{-k}.$$

More generally, a rational function

$$f(z) = \frac{p_n(z)}{q_m(z)}, \qquad \deg(p) = n,\ \deg(q) = m$$

has order $m - n$ at infinity, giving the pleasant relation

$$\sum_{c \in \mathbf{C} \cup \infty} \operatorname{ord}_c(f) = 0.$$

An entire transcendental function $f$ has an essential singularity at $\infty$ since

$$g(\zeta) = f(1/\zeta) = \sum_{n=0}^{\infty} a_n \zeta^{-n}.$$

And the principal part of any Laurent expansion has a removable singularity at $\infty$. (This finishes exercise 5(a); in 5(b) the singularity at $\infty$ is not isolated since $g(\zeta) = f(1/\zeta)$ isn't analytic in any punctured disk about 0.)

What's going on here is that the Riemann sphere, while globally distinct from $\mathbf{C}$, is indistinguishable from $\mathbf{C}$ in the small. To study a function at $\infty$, we use the mapping

$$z \longmapsto 1/z \stackrel{\text{call}}{=} \zeta$$

to take a neighborhood of $\infty$ homeomorphically to a neighborhood of 0, since we understand how to analyze singularities at 0. The notion we are tiptoeing around here is that of a *manifold*, loosely a topological space that looks Euclidean in the small. More generally than studying functions at infinity, if $f$ has a nonisolated singularity at $c$ and some mapping $\varphi$ takes a neighborhood of a point $p \in \mathbf{C}$ analytically and homeomorphically to a neighborhood of $c$, then the singularity of $f$ at $c$ is of the same type as the singularity of $f \circ \varphi$ at $p$. Proving this is an exercise in manipulating Laurent series that cites the yet-unproven fact that

$$\varphi(\zeta) - c = b_1(\zeta - p) + \cdots \ \text{(where } b_1 \neq 0\text{)} \quad \text{for } \zeta \text{ nearZ } p.$$

Changing variables like this makes exercise 5(c) easy at $z = 1/n$ where $n$ is a nonzero integer. Let $z = 1/(\zeta + n)$ (so $\zeta = 1/z - n$) to study $\sin(\pi/z)$ at $1/n$ by studying $\sin \pi(\zeta + n\pi)$ at 0. The singularity at 0 is nonisolated.

## 4. More Function-Theoretic results

With the results here, a more general version of Casorati-Weierstrass is easy.

**Theorem 4.1** (Casorati–Weierstrass Theorem, version 2)**.** *If $f$ has an essential singularity at $\infty$ then for any $R > 0$, the set*

$$\{f(z) : |z| > R\}$$

*is dense in* $\mathbf{C}$.

*Proof.* Consider the Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n, \quad \text{for large } |z|.$$

This breaks into a principal part and an entire transcendental function. The principal part has absolute value less than $\varepsilon/2$ for $|z|$ large enough, while the entire transcendental function gets within $\varepsilon/2$ of any $c \in \mathbf{C}$ for infinitely many large $|z|$ by the previous version of Casorati-Weierstrass. The result follows. □

There is nothing special about infinity, actually. The final version of Casorati-Weierstrass is

**Theorem 4.2** (Casorati–Weierstras Theorem version 3)**.** *If $f$ has an essential singularity at $c \in \mathbf{C} \cup \infty$ then for any neighborhood $N$ of $c$, the set*

$$f(N \setminus \{c\})$$

*is dense in* $\mathbf{C}$.

*Proof.* The result is already established if $c = \infty$. If$c \in \mathbf{C}$ instead then let $g(z) = f(z + c)$, which has an essential singularity at 0, and then let $h(z) = g(1/z)$, which has an essential singularity at $\infty$. Since $h$ takes large inputs to a dense set of outputs, $g$ takes inputs near 0 to a dense set of outputs, and so $f$ takes inputs near $c$ to a dense set of outputs. □

The summary theorem about singularities is called

**Theorem 4.3** (Riemann's Theorem)**.** *Let $f$ have an isolated singularity at the point $c \in \mathbf{C} \cup \infty$. The singularity is*

- *removable if and only if $f$ is bounded near $c$,*
- *a pole if and only if $|f(z)| \to +\infty$ as $z \to c$,*
- *essential if and only if $f$ behaves in any other fashion.*

What makes this theorem satisfying is that it perfectly matches up the various series-based descriptions of $f$ about $c$ with the various behavioral (i.e., function-theoretic) descriptions of $f$ near $c$.

*Proof.* Consider the Laurent series of $f$ about $c$. If $c \in \mathbf{C}$, the Laurent series is

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - c)^n,$$

while if $c = \infty$ then the Laurent series is

$$f(z) = \sum b_n \zeta^n, \quad \zeta = 1/z \text{ and } b_n = a_{-n}.$$

If the singularity at $c$ is removable than $a_n = 0$ (or $b_n = 0$) for all $n < 0$, and so $f(z) \to a_0$ (or $b_0$) as $z \to c$.

If the singularity at $c$ is a pole of order $N > 0$ then $f(z) = a_{(-N)}(z-c)^{-N}(1 + a_{(-N+1)}(z-c) + \dots)$ or $f(z) = b_{(-N)}\zeta^{-N}(1 + b_{(-N+1)}\zeta + \dots)$, which goes to $\infty$ as $z \to c$ or $\zeta \to 0$.

If the singularity is essential then $f$ is neither bounded nor uniformly large, by the Casorati-Weierstrass theorem. Thus the three implications $\implies$ are proved. And since the three behaviors are exclusive and exhaustive, the three implications $\impliedby$ follow. $\square$