ZETA AT NEGATIVE ODD INTEGERS, *A LA* EULER

This writeup sketches (you may need to supply details) an argument due to Euler that partially establishes the the functional equation of $\zeta(s)$. In particular, it leads to the formula

$$\zeta(1-k) = -\frac{B_k}{k} \quad \text{for even } k \geq 2.$$

Let t be a formal variable. Starting from the identity

$$t + t^2 + t^3 + t^4 + \cdots = (t - t^2 + t^3 - t^4 + \cdots) + 2(t^2 + t^4 + t^6 + t^8 + \cdots),$$

apply the operator $t\frac{d}{dt}$ (i.e., differentiation and then multiplication by t) n times to get

$$\begin{aligned} &1^n t + 2^n t^2 + 3^n t^3 + 4^n t^4 + \cdots \\ &\qquad = \left(t\frac{d}{dt}\right)^n \left(\frac{t}{1+t}\right) + 2^{n+1}(1^n t^2 + 2^n t^4 + 3^n t^6 + 4^n t^8 + \cdots). \end{aligned}$$

Formally, when $t = 1$ this is

$$\zeta(-n) = \left[\left(t\frac{d}{dt}\right)^n \left(\frac{t}{1+t}\right)\right]_{t=1} + 2^{n+1}\zeta(-n),$$

giving a heuristic value for $\zeta(-n)$,

$$\zeta(-n) = (1 - 2^{n+1})^{-1} \cdot \left[\left(t\frac{d}{dt}\right)^n \left(\frac{t}{1+t}\right)\right]_{t=1}.$$

Thus for example, according to Euler,

$$1 + 1 + 1 + 1 + \cdots = -\frac{1}{2} \quad \text{and} \quad 1 + 2 + 3 + 4 + \cdots = -\frac{1}{12}.$$

Next, let $t = e^X$ and note that $t\frac{d}{dt} = \frac{d}{dX}$. Now we have

$$(1 - 2^{n+1})\zeta(-n) = \left[\frac{d^n}{dX^n}\left(\frac{e^X}{e^X + 1}\right)\right]_{X=0} \quad \text{for } n \in \mathbb{N},$$

giving the Taylor series

$$\frac{e^X}{e^X + 1} = \sum_{n=0}^{\infty} \frac{(1 - 2^{n+1})\zeta(-n)}{n!} X^n.$$

Thus the function of a complex variable

$$F(z) = \frac{e^{2\pi iz}}{e^{2\pi iz} + 1} = \sum_{n=0}^{\infty} \frac{(1 - 2^{n+1})\zeta(-n)(2\pi i)^n}{n!} z^n$$

generates (in the sense of generating function) the values $\zeta(-n)$ for all natural numbers n.

Let $G(z) = \pi \cot \pi z$. Recall that like F, G is both a fractional linear function of $e^{2\pi i z}$ and a generating function for zeta,

$$G(z) = \pi i \frac{e^{2\pi i z} + 1}{e^{2\pi i z} - 1} = \frac{1}{z} - 2\sum_{k=1}^{\infty} \zeta(2k) z^{2k-1}.$$

The fractional linear forms of F and G show that there must be a relation between them, and therefore between $\zeta(1-k)$ and $\zeta(k)$ for even $k \geq 2$. Indeed, the relation between F and G is

$$\frac{1}{\pi i}(G(z) - 2G(2z)) = -F(z) + F(-z),$$

and so equating coefficients gives

$$\zeta(1-k) = \frac{2\Gamma(k)}{(2\pi i)^k}\zeta(k) \quad \text{for even } k \geq 2.$$

The zeta-value that we have computed previously,

$$\zeta(k) = -\frac{1}{2} \cdot \frac{(2\pi i)^k B_k}{k!} \quad \text{for even } k \geq 2$$

now gives the identity promised at the beginning of the writeup,

$$\zeta(1-k) = -\frac{B_k}{k} \quad \text{for even } k \geq 2.$$

We want to symmetrize the previous two displays. A calculation shows that

$$\frac{\pi^{-\frac{1-k}{2}}\Gamma\left(\frac{1-k}{2}\right)}{\pi^{-\frac{k}{2}}\Gamma\left(\frac{k}{2}\right)} = \frac{1}{2} \cdot \frac{(2\pi i)^k}{(k-1)!}.$$

It follows that

$$\pi^{-k/2}\Gamma(\tfrac{k}{2})\zeta(k) = \pi^{-(1-k)/2}\Gamma(\tfrac{1-k}{2})\zeta(1-k) \quad \text{for even } k \geq 2.$$

This is a partial version of the *functional equation* of the zeta function. The full functional equation is

$$\pi^{-s/2}\Gamma(\tfrac{s}{2})\zeta(s) = \pi^{-(1-s)/2}\Gamma(\tfrac{1-s}{2})\zeta(1-s) \quad \text{for all } s \in \mathbb{C}.$$

This is usually written

$$\xi(s) = \xi(1-s) \quad \text{for all } s \in \mathbb{C}$$

where

$$\xi(s) = \pi^{-s/2}\Gamma(\tfrac{s}{2})\zeta(s).$$

Euler's ideas here can be turned into a rigorous proof of the meromorphic continuation and the functional equation of ζ, but the proof that we may give later in the course will not follow these lines. One comment to make for now is that the zeta function has an *Euler factorization*,

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1} \quad \text{for } \mathrm{Re}(s) > 1,$$

where the product is taken over all primes p. From a modern perspective, each Euler factor $(1-p^{-s})^{-1}$ can be interpreted as an integral, taken in an environment particular to the prime p, and the extra factor $\pi^{-s/2}\Gamma(s/2)$ in the symmetrized function $\xi(s)$ is the same integral, but taken in an environment particular to a conceptually infinite prime.