

A SERIES REPRESENTATION OF THE COTANGENT

This writeup establishes an equality of meromorphic functions,

$$\begin{aligned}\pi \cot \pi z &= \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z-n} + \frac{1}{z+n} \right) \\ &= \frac{1}{z} + 2z \sum_{n=1}^{\infty} \frac{1}{z^2 - n^2}.\end{aligned}$$

The function $\pi \cot \pi z$ (for nonintegers $z \in \mathbb{C}$) is analytic and \mathbb{Z} -periodic. Near $z = 0$ we have

$$\pi \cot \pi z \sim \pi \frac{1}{\pi z} = \frac{1}{z},$$

so that $\pi \cot \pi z$ is also meromorphic at 0, having a simple pole there with residue 1. By \mathbb{Z} -periodicity, the same holds at each integer n . Thus, a naïve first attempt to imitate $\pi \cot \pi z$ by a series is

$$\sum_{n \in \mathbb{Z}} \frac{1}{z-n}.$$

However, the n th term of this series is $\mathcal{O}(1/n)$, so that the series is not even summable. One can fix this problem by modifying the terms to obtain the series

$$\frac{1}{z} + \sum_{n \neq 0} \left(\frac{1}{z-n} + \frac{1}{n} \right).$$

Now the n th term is

$$\frac{1}{z-n} + \frac{1}{n} = \frac{z}{(z-n)n} = \mathcal{O}\left(\frac{1}{n^2}\right),$$

and so the new series *is* summable. In fact, this calculation shows that the new series is absolutely summable, so that its terms can be rearranged. In particular, pairing the terms for n and $-n$ gives

$$\begin{aligned}\frac{1}{z-n} + \frac{1}{n} + \frac{1}{z+n} - \frac{1}{n} &= \frac{1}{z-n} + \frac{1}{z+n} \\ &= \frac{2z}{z^2 - n^2},\end{aligned}$$

and these are the terms of the series that we began with, in both of its forms. So at least that series converges absolutely for any noninteger $z \in \mathbb{C}$.

The convergence is gained at a cost, but the cost is only apparent. With its terms corrected, when our series is evaluated at z and then at $z+1$ its symmetric partial sums are

$$\varphi_n(z) = \frac{1}{z} + \sum_{1 \leq |j| \leq n} \left(\frac{1}{z-j} + \frac{1}{j} \right) = \sum_{j=-n}^n \frac{1}{z-j}$$

and

$$\varphi_n(z+1) = \frac{1}{z+1} + \sum_{1 \leq |j| \leq n} \left(\frac{1}{z+1-j} + \frac{1}{j} \right) = \sum_{j=-n}^n \frac{1}{z+1-j},$$

and these are not identical termwise. However, their difference nearly cancels,

$$\varphi_n(z+1) - \varphi_n(z) = \frac{1}{z+n+1} - \frac{1}{z-n},$$

and the remaining two terms go to 0 for any z as n grows. Thus $\lim_n \varphi_n(z)$ is indeed \mathbb{Z} -periodic. Below we will give a second proof of its periodicity.

All of this said, the series that this writeup begins with (in either of its forms) is not a Laurent series, and so part of the task here is to show that it defines a meromorphic function at all.

To show that the sum is meromorphic, recall a result from a previous writeup: *Let Ω be a region in \mathbb{C} . Consider a sequence of differentiable functions on Ω ,*

$$\{\varphi_0, \varphi_1, \varphi_2, \dots\} : \Omega \longrightarrow \mathbb{C}.$$

Suppose that the sequence converges on Ω to a limit function

$$\varphi : \Omega \longrightarrow \mathbb{C}$$

and that the convergence is uniform on compact subsets of Ω . Then

- (1) *The limit function φ is differentiable.*
- (2) *The sequence $\{\varphi'_n\}$ of derivatives converges on Ω to the derivative φ' of the limit function.*
- (3) *This convergence is also uniform on compact subsets of Ω .*

To apply the result here, let $\Omega = \mathbb{C} - \mathbb{Z}$, a region in \mathbb{C} . Define

$$\varphi_n : \Omega \longrightarrow \mathbb{C}, \quad \varphi_n(z) = \frac{1}{z} + \sum_{j=1}^n \left(\frac{1}{z-j} + \frac{1}{z+j} \right), \quad n = 0, 1, 2, \dots$$

This is the sequence of partial sums of

$$\varphi : \Omega \longrightarrow \mathbb{C}, \quad \varphi(z) = \frac{1}{z} + \sum_{j=1}^{\infty} \left(\frac{1}{z-j} + \frac{1}{z+j} \right).$$

Consider any $z \in \Omega$. For all $j > \sqrt{2}|z|$, the reverse triangle inequality gives

$$|z^2 - j^2| \geq j^2 - |z|^2 > j^2 - j^2/2 = j^2/2,$$

and so

$$\left| \frac{1}{z^2 - j^2} \right| < \frac{2}{j^2}.$$

This shows that the partial sums

$$\varphi_n(z) = \frac{1}{z} + 2z \sum_{j=1}^n \frac{1}{z^2 - j^2}$$

converge absolutely. Consequently, they converge to the limit function

$$\varphi(z) = \frac{1}{z} + 2z \sum_{j=1}^{\infty} \frac{1}{z^2 - j^2}.$$

We need to show that the convergence is uniform on compact subsets of Ω . Let K be such a subset, and let $\varepsilon > 0$ be given. There is a uniform bound $b > 0$ on the absolute values $|z|$ for all $z \in K$. Also, there a starting index n_0 such that for any $n > n_0$,

$$\sum_{j=n+1}^{\infty} \frac{1}{j^2} < \frac{\varepsilon}{4b}.$$

Consider any n such that $n > n_0$ and also $n > \sqrt{2b}$. For such n and for all $z \in K$,

$$|\varphi(z) - \varphi_n(z)| = \left| 2z \sum_{j=n+1}^{\infty} \frac{1}{z^2 - j^2} \right| \leq 2b \sum_{j=n+1}^{\infty} \left| \frac{1}{z^2 - j^2} \right| \leq 2b \sum_{j=n+1}^{\infty} \frac{2}{j^2} < \varepsilon.$$

This shows that the convergence of $\{\varphi_n\}$ to φ on Ω is uniform on compact subsets.

By the result, the limit function can be differentiated termwise. Now that we no longer need the symbol n to index partial sums, we return to the more natural notation of using it as sum-index,

$$\varphi(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z-n} + \frac{1}{z+n} \right) = \frac{1}{z} + 2z \sum_{n=1}^{\infty} \frac{1}{z^2 - n^2},$$

and

$$\varphi'(z) = - \sum_{n \in \mathbb{Z}} \frac{1}{(z-n)^2}.$$

The second series for φ shows that it is odd, and the series for φ' shows that it is even. The convergence of φ' is again absolute, and so φ' is \mathbb{Z} -periodic by a calculation that rearranges terms,

$$\varphi'(z+m) = - \sum_{n \in \mathbb{Z}} \frac{1}{(z+m-n)^2} = - \sum_{n' \in \mathbb{Z}} \frac{1}{(z-n')^2} \quad \text{where } n' = n - m.$$

It follows that

$$(\varphi(z+1) - \varphi(z))' = \varphi'(z+1) - \varphi'(z) = \varphi'(z) - \varphi'(z) = 0,$$

so that

$$\varphi(z+1) - \varphi(z) = c \quad \text{for some constant } c.$$

To show (for a second time) that φ is \mathbb{Z} -periodic, we need to show that $c = 0$. But in particular,

$$c = \varphi(1/2) - \varphi(-1/2) = 2\varphi(1/2) \quad \text{since } \varphi \text{ is odd,}$$

and so it suffices to show that $\varphi(1/2) = 0$. Inspect it,

$$\varphi(1/2) = 2 + \sum_{n=1}^{\infty} \frac{1}{\frac{1}{4} - n^2} = 2 - \sum_{n=1}^{\infty} \left(\frac{1}{n - \frac{1}{2}} - \frac{1}{n + \frac{1}{2}} \right).$$

The sum telescopes to 2, giving the desired result.

The argument so far shows that the function $\varphi(z) - 1/z$ is also analytic at $z = 0$. Therefore φ itself is meromorphic at 0, having a simple pole there with residue 1. By the \mathbb{Z} -periodicity, the same holds at each integer n . This matches the behavior of $\pi \cot \pi z$. Thus the difference $\pi \cot \pi z - \varphi(z)$ is entire. We want to show that it is the zero function.

The first step is to show that the difference is bounded, making it constant by Liouville's theorem. Since the difference is \mathbb{Z} -periodic in the x -direction, it suffices

to show that is bounded as $|y| \rightarrow \infty$, and for this it suffices to show that each of $\pi \cot \pi z$ and $\varphi(z)$ is individually bounded as $|y| \rightarrow \infty$. Compute first that

$$\pi \cot \pi z = \pi i \frac{e^{\pi iz} + e^{-\pi iz}}{e^{\pi iz} - e^{-\pi iz}} = \pi i \frac{e^{2\pi iz} + 1}{e^{2\pi iz} - 1} = \pi i + \frac{2\pi i}{e^{2\pi iz} - 1}.$$

Also $|e^{2\pi iz}| = e^{-2\pi y}$, so $\lim_{y \rightarrow +\infty} \pi \cot \pi z = -\pi i$ and $\lim_{y \rightarrow -\infty} \pi \cot \pi z = \pi i$. On the other hand, suppose now that $z = x + iy$ where $0 \leq x < 1$ and $|y| > 1$. Then we have the inequalities $|y| \leq |z| \leq |y| + 1$ and

$$|z^2 - n^2| = |x^2 - y^2 - n^2 + 2ixy| \geq y^2 + n^2 - x^2 \geq y^2 + n^2 - 1.$$

It follows that

$$|\varphi(z)| \leq \frac{1}{|y|} + 2(|y| + 1) \sum_{n=1}^{\infty} \frac{1}{y^2 + n^2 - 1}.$$

Let $\eta = \lfloor |y| \rfloor$. Then

$$\sum_{n=1}^{\infty} \frac{1}{y^2 + n^2 - 1} = \sum_{m=0}^{\infty} \sum_{r=1}^{\eta} \frac{1}{y^2 + (m\eta + r)^2 - 1},$$

and for each $m \geq 0$,

$$\sum_{r=1}^{\eta} \frac{1}{y^2 + (m\eta + r)^2 - 1} \leq \frac{\eta}{\eta^2 + (m\eta)^2} = \frac{1}{\eta(1 + m^2)}.$$

This shows that

$$|\varphi(z)| \leq \frac{1}{|y|} + 2 \frac{|y| + 1}{\lfloor |y| \rfloor} \sum_{m=0}^{\infty} \frac{1}{1 + m^2},$$

and so $\varphi(z)$ is bounded as $|y| \rightarrow \infty$ as well.

Thus $\pi \cot \pi z - \varphi(z)$ is constant. To see that the constant is 0, set $z = 1/2$. From before, $\varphi(1/2) = 0$. But also $\pi \cot \pi/2 = 0$, giving the result. And further, termwise differentiability of

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z - n} + \frac{1}{z + n} \right)$$

also gives

$$\pi^2 \csc^2 \pi z = \sum_{n=-\infty}^{\infty} \frac{1}{(z - n)^2},$$

now with absolute convergence even without grouping the terms for $\pm n$.

As an application, we compare the power series expansions about $z = 0$ of the two now-known-to-be-equal functions

$$z\varphi(z) \quad \text{and} \quad \pi z \cot \pi z.$$

For the first expansion, compute that for $|z| < 1$,

$$\begin{aligned} z\varphi(z) &= 1 + 2z^2 \sum_{n=1}^{\infty} \frac{1}{z^2 - n^2} = 1 - 2z^2 \sum_{n=1}^{\infty} \frac{1}{n^2} \cdot \frac{1}{1 - z^2/n^2} \\ &= 1 - 2z^2 \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{k=0}^{\infty} \left(\frac{z}{n}\right)^{2k} = 1 - 2 \sum_{k=0}^{\infty} z^{2k+2} \sum_{n=1}^{\infty} \frac{1}{n^{2k+2}} \\ &= 1 - 2 \sum_{\text{even } k \geq 2} \zeta(k) z^k. \end{aligned}$$

That is, $z\varphi(z)$ is a generating function for the Euler–Riemann zeta function $\zeta(k)$ at positive even values of k . On the other hand, the second expansion is essentially a generating function for the Bernoulli numbers. Again for $|z| < 1$,

$$\begin{aligned} \pi z \cot \pi z &= \pi i z + \frac{2\pi i z}{e^{2\pi i z} - 1} = \pi i z + \sum_{k=0}^{\infty} \frac{B_k}{k!} (2\pi i z)^k \\ &= 1 + \sum_{\text{even } k \geq 2} \frac{(2\pi i)^k B_k}{k!} z^k. \end{aligned}$$

Comparing the two shows expansions gives Euler’s famous formula,

$$\zeta(k) = -\frac{1}{2} \cdot \frac{(2\pi i)^k B_k}{k!} \quad \text{for all even } k \geq 2.$$

In particular, this formula combines with the values $B_2 = 1/6$, $B_4 = -1/30$, $B_6 = 1/42$ to give

$$\zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}, \quad \zeta(6) = \frac{\pi^6}{945}.$$

Euler’s formula for $\zeta(k)$ (even $k \geq 2$) can also be obtained by contour integration techniques, as in our text. The idea is that since

$$\pi \cot \pi z = \frac{1}{z} + \sum_{\text{even } k \geq 2} \frac{(2\pi i)^k B_k}{k!} z^{k-1},$$

it follows that for any even $k \geq 2$,

$$\text{Res}_{z=0} \left(\frac{\pi \cot \pi z}{z^k} \right) = \frac{(2\pi i)^k B_k}{k!}.$$

By contour integration,

$$\frac{(2\pi i)^k B_k}{k!} + 2\zeta(k) = 0,$$

and Euler’s formula follows immediately.

Since $\pi \cot \pi z$ is \mathbb{Z} -periodic it also has a Fourier series expansion. This is not the same thing as its Laurent series expansion. Instead, if $z = x + iy$ with $y > 0$ then $|e^{2\pi i z}| = e^{-2\pi y} < 1$, and so

$$\begin{aligned} \pi \cot \pi z &= \pi i + \frac{2\pi i}{e^{2\pi i z} - 1} \\ &= \pi i - 2\pi i \sum_{n=0}^{\infty} q^n, \quad \text{where } q = e^{2\pi i z}. \end{aligned}$$