CIRCLES AND LINES

Working in the environment of the complex projective line $\mathbb{P}^1(\mathbb{C})$ rather than the Riemann sphere $\widehat{\mathbb{C}}$ gives an elegant uniform description of lines and circles, and an immediate case-free argument that they are preserved under fractional linear transformations.

The first idea is that *every real-projective class of 2-by-2 nonpositive hermitian (self-adjoint) matrices describes a circle, a line, or a point, and conversely.* Indeed, let m be a nonzero 2-by-2 nonpositive hermitian matrix,

$$m = \begin{bmatrix} a & -b \\ -\overline{b} & d \end{bmatrix}, \quad m \neq 0,\ a, d \in \mathbb{R},\ b \in \mathbb{C},\ ad - |b|^2 \leq 0.$$

Consider nonzero column vectors

$$v = \begin{bmatrix} z \\ w \end{bmatrix}, \quad v \neq 0,\ z, w \in \mathbb{C},$$

and consider the equation (in which "$*$" is the transpose-conjugate operator)

$$v^* m v = 0.$$

In coordinates, this equation is

$$\begin{bmatrix} \overline{z} & \overline{w} \end{bmatrix} \begin{bmatrix} a & -b \\ -\overline{b} & d \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix} = 0,$$

or

$$a|z|^2 - 2\mathrm{Re}(\overline{b} z \overline{w}) + d|w|^2 = 0.$$

For any nonzero $r \in \mathbb{R}$, the scaled matrix rm is again nonpositive hermitian, and it gives rise to the r-scaled version of the equation shown in the previous three displays, with no effect on the set of solution-vectors v. Similarly, if a nonzero vector v satisfies the equation then for any nonzero $\lambda \in \mathbb{C}$, so does the scaled vector λv. That is, we can view the matrix m defining the equation as real-projective and the vector variable v as complex-projective.

If $a \neq 0$ then we can normalize to $a = 1$. The vector solutions with $w \neq 0$ can be normalized to $w = 1$, giving the finite points z of the Riemann sphere such that

$$|z|^2 - \overline{b} z - b \overline{z} + d = 0,$$

or

$$|z - b|^2 = |b|^2 - d$$

Note that $|b|^2 - d \geq 0$ since m is nonpositive. So when $a \neq 0$, the affine solutions of the equation form a circle or reduce to a single point. The infinite point $\infty = [1, 0]$ of the Riemann sphere is not a solution. In sum, when $a \neq 0$, the equation describes an affine circle or an affine point.

Now suppose that $a = 0$. This time the equation is

$$-2\mathrm{Re}(\overline{b} z \overline{w}) + d|w|^2 = 0.$$

The infinite point $\infty = [1, 0]$ of the Riemann sphere is a solution. The affine solutions are the finite points z of the Riemann sphere such that

$$-\overline{b}z - b\overline{z} + d = 0.$$

Let $b = (\alpha + i\beta)/2$. Then we have

$$\alpha x + \beta y = d.$$

If $b \neq 0$ then at least one of α and β is nonzero, and this equation describes a line. On the other hand, if $b = 0$ then necessarily $d \neq 0$ (since $m \neq 0$), so there are no affine solutions and the projective solution set is the infinite point. In sum, when $a = 0$, the equation describes a line or the point at infinity.

These arguments have shown that every real-projective class of 2-by-2 nonpositive hermitian matrices describes a circle or a line or a point. The converse is shown by running the arguments in the other direction. The degenerate solution sets consisting of a single point (affine or at infinity) can be eliminated from the discussion by making m negative rather than nonpositive. The geometric idea that circles and lines are the same thing on the Riemann sphere is manifested here in the algebraic idea that circles and lines have a uniform projective description.

Now note that if m is a 2-by-2 nonpositive hermitian matrix, and g is any invertible 2-by-2 complex matrix whatsover, then g^*mg is again nonpositive hermitian. Note also that for any vector $v \in \mathbb{C}^2$,

$$(gv)^* m(gv) = v^*(g^*mg)v.$$

In words, this says that

gv lies on the circle or line described by m
if and only if
v lies on the circle or line described by g^*mg.

As an immediate consequence, fractional linear transformations preserve circles and lines.