CIRCLES AND LINES

Working in the environment of the complex projective line $\mathbb{P}^1(\mathbb{C})$ rather than the Riemann sphere $\widehat{\mathbb{C}}$ gives a clear description of lines and circles, with the immediate consequence that they are preserved under fractional linear transformations.

The first idea is that every real-projective class of 2-by-2 nonpositive hermitian (self-adjoint) matrices describes a circle, a line, or a point, and conversely. Indeed, let m be a nonzero 2-by-2 nonpositive hermitian matrix,

$$m = \begin{bmatrix} a & -b \\ -\overline{b} & d \end{bmatrix}, \quad m \neq 0, \ a, d \in \mathbb{R}, \ b \in \mathbb{C}, \ ad - |b|^2 \le 0.$$

Consider nonzero column vectors

$$v = \begin{bmatrix} z \\ w \end{bmatrix}, \quad v \neq 0, \ z, w \in \mathbb{C},$$

and consider the equation (in which "*" is the transpose-conjugate operator)

$$v^*mv = 0.$$

For any nonzero $r \in \mathbb{R}$, the scaled matrix rm is again nonpositive hermitian, and it gives rise to the *r*-scaled version of this equation, with the same set of solutionvectors v. Similarly, if a nonzero vector v satisfies this equation then for any nonzero $\lambda \in \mathbb{C}$, so does the scaled vector λv . That is, we can view the matrix m defining this equation as real-projective and the vector variable v as complex-projective. In coordinates, the equation is

$$\begin{bmatrix} \overline{z} & \overline{w} \end{bmatrix} \begin{bmatrix} a & -b \\ -\overline{b} & d \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix} = 0,$$

or

$$a|z|^2 - 2\operatorname{Re}(\overline{b}z\overline{w}) + d|w|^2 = 0.$$

If $a \neq 0$ then we scale to a = 1. And we scale the vector solutions with $w \neq 0$ to w = 1, giving the finite points z of the Riemann sphere such that

$$|z|^2 - \overline{b}z - b\overline{z} + d = 0$$

or

$$|z - b|^2 = |b|^2 - d$$

Here $|b|^2 - d \ge 0$ since *m* is nonpositive. So when $a \ne 0$, the affine solutions of the equation form a circle or reduce to a single point. The infinite point $\infty = [1,0]$ of the Riemann sphere is not a solution. Overall, when $a \ne 0$ the equation describes an affine circle or an affine point.

Now suppose that a = 0. This time the equation is

$$-2\operatorname{Re}(\overline{b}z\overline{w}) + d|w|^2 = 0.$$

The infinite point $\infty = [1,0]$ of the Riemann sphere is a solution. The affine solutions are the finite points z of the Riemann sphere such that

$$-\overline{b}z - b\overline{z} + d = 0$$

Let $b = (\alpha + i\beta)/2$. Then we have

$$\alpha x + \beta y = d.$$

If $b \neq 0$ then at least one of α and β is nonzero, and this equation describes a line. On the other hand, if b = 0 then necessarily $d \neq 0$ (since $m \neq 0$), so there are no affine solutions and the projective solution set is the infinite point. Overall, when a = 0 the equation describes a line or the point at infinity.

These arguments have shown that every real-projective class of 2-by-2 nonpositive hermitian matrices describes a circle or a line or a point. The converse is shown by running the arguments in the other direction. The degenerate solution sets consisting of a single point (affine or at infinity) can be eliminated from this discussion by making m negative rather than nonpositive. The geometric idea that circles and lines are the same thing on the Riemann sphere is expressed here in the algebraic idea that circles and lines have a uniform projective description.

Now the main point of this writeup is very quick. If m is a 2-by-2 nonpositive hermitian matrix, and g is any invertible 2-by-2 complex matrix whatsover, then $g^{-*}mg^{-1}$ is again nonpositive hermitian. For any vector $v \in \mathbb{C}^2$,

$$v^*mv = (gv)^*(g^{-*}mg^{-1})(gv),$$

and so

$$v^*mv = 0 \iff (gv)^*(g^{-*}mg^{-1})(gv) = 0.$$

In words, this says that

v lies on the circle or line described by m if and only if gv lies on the circle or line described by $g^{-*}mg^{-1}$.

Thus fractional linear transformations preserve circles and lines.