CAUCHY'S THEOREM FOR SIMPLE CURVES

Let Ω be a region in $\mathbb{C}$ and let $f : \Omega \longrightarrow \mathbb{C}$ be differentiable. Cauchy's theorem for simple curves says that if γ is a simple closed rectifiable curve in Ω whose interior lies in Ω then $\int_\gamma f(z)\,\mathrm{d}z = 0$. We establish the theorem for triangles, then simple polygons, then polygons, then simple closed rectifiable curves.

Contents

1. Cauchy's Theorem for Triangles

Let Ω be a region, let $f : \Omega \longrightarrow \mathbb{C}$ be differentiable, and let $\mathbb{T}$ be a triangle in Ω. Here $\mathbb{T}$ is not only three line segments, but also the region that they surround. The three segments, traversed counterclockwise, are denoted T. That is, $T = \partial\mathbb{T}$. Cauchy's Theorem for triangles states that

$$\int_T f(z)\,\mathrm{d}z = 0.$$

Bisect each side of T to get four counterclockwise triangles $T_1^j = \partial\mathbb{T}_1^j$ for $j = 1, \ldots, 4$. Then

$$\left|\int_T f(z)\,\mathrm{d}z\right| = \left|\sum_{j=1}^{4}\int_{T_1^j} f(z)\,\mathrm{d}z\right| \le \sum_{j=1}^{4}\left|\int_{T_1^j} f(z)\,\mathrm{d}z\right| \le 4\left|\int_{T_1} f(z)\,\mathrm{d}z\right|,$$

where T_1 is one of the triangles T_1^j. Iterating the argument shows that for any $n > 0$ we have

$$\left|\int_T f(z)\,\mathrm{d}z\right| \le 4^n\left|\int_{T_n} f(z)\,\mathrm{d}z\right|, \tag{1}$$

where $T_n = \partial\mathbb{T}_n$ and $\mathbb{T} \supset \mathbb{T}_1 \supset \cdots \supset \mathbb{T}_n$, each triangle's sides being half as long as those of the triangle before it.

The intersection of all the solid triangles is a single point,

$$\bigcap_{n\ge 1} \mathbb{T}_n = \{p\}.$$

Indeed, because the triangle diameters shrink by a factor of two at each generation, the intersection can't be more than one point. And because no finite intersection of $\mathbb{T}_n$'s is empty, the infinite intersection isn't empty either because $\mathbb{T}$ is compact.

For convenience we may assume that $p = 0$ and that $f(0) = 0$. The condition that f is differentiable at 0 is that for some constant $c \in \mathbb{C}$,

$$f(z) = cz + o(z).$$

Consequently,

$$\int_{T_n} f(z)\,\mathrm{d}z = c\int_{T_n} z\,\mathrm{d}z + \int_{T_n} o(z)\,\mathrm{d}z.$$

Because z has antiderivative z^2 and T_n is closed, the first integral is 0, and so in fact

$$\int_{T_n} f(z)\,\mathrm{d}z = \int_{T_n} o(z)\,\mathrm{d}z.$$

Let $\varepsilon > 0$ be given. Because the triangles $\{T_n\}$ are shrinking to 0, we have $o(z) \leq \varepsilon|z|$ for all $z \in T_n$ as soon as n is large enough. For such n,

$$\left|\int_{T_n} f(z)\,\mathrm{d}z\right| \leq \varepsilon \sup\{|z| : z \in T_n\} \cdot \operatorname{length}(T_n) \leq \varepsilon \cdot (\operatorname{length}(T_n))^2.$$

But $\operatorname{length}(T_n) = \operatorname{length}(T)/2^n$. Therefore, for large enough n,

$$\left|\int_{T_n} f(z)\,\mathrm{d}z\right| \leq \varepsilon \cdot (\operatorname{length}(T))^2/4^n.$$

Combine inequality (1) with these results to get

$$\left|\int_{T} f(z)\,\mathrm{d}z\right| \leq 4^n \left|\int_{T_n} f(z)\,\mathrm{d}z\right| \leq \varepsilon \cdot (\operatorname{length}(T))^2.$$

Because $\varepsilon > 0$ is arbitrary and $\operatorname{length}(T)$ is finite, the desired result follows,

$$\int_{T} f(z)\,\mathrm{d}z = 0.$$

2. Cauchy's Theorem for Simple Polygons

Let Ω be a region, let $f : \Omega \longrightarrow \mathbb{C}$ be differentiable, and let $\mathbb{P}$ be a simple polygon in Ω. Here $\mathbb{P}$ is not only the boundary segments, but also the region that they surround. The segments, traversed counterclockwise, are denoted P. That is, $P = \partial\mathbb{P}$. To say that the polygon is *simple* is to say that the only intersection points of the segments are each segment's endpoint and the start-point of the next segment, and the last segment's endpoint and the start-point of the first segment. Cauchy's Theorem for simple polygons states that

$$\int_{P} f(z)\,\mathrm{d}z = 0.$$

This can be shown by induction on the number of polygon vertices, with the triangle as the base case. To do so, we show that some pair of polygon vertices can see each other, in the sense that the segment joining them lies entirely inside the polygon; add that segment and then cut along it to get two simple polygons, each of which has fewer vertices than the original. So, assume that the simple polygon P has more than three vertices. Let B be an outward-pointing vertex of the polygon P; such a vertex exists because the sum 2π of the external angles at the vertices is positive. Let A and C be its neighboring vertices, with the interior of the polygon to the left as we move from A to B to C. If A and C can see each other then we are done. Otherwise, consider the line through B parallel to AC; start moving it from B toward AC, keeping it parallel to AC. There is a first moment where the segment strictly between AB and BC of the moving line meets the polygon, and at that moment the intersection contains a vertex D such that B and D see each other.

3. Cauchy's Theorem for Polygons

Given a non-simple polygon, we may add vertices where edges cross or share a segment, eliminate back-and-forth edge traversals, and be left with finitely many simple polygons. Specifically, if after adding vertices as just described, if the vertex list contains a section $\dots, a, b, c, b, d \dots$ then replace it by $\dots a, b, d, \dots$, and if the vertex list takes the form $a_1, \dots, a_k, b, c_1, \dots, c_\ell, b, d_1, \dots, d_n, a_1$ with $\ell \geq 2$ then replace it by $a_1, \dots, a_k, b, d_1, \dots, d_n, a_1$ and $b, c_1, \dots, c_\ell, b$; repeat this process until obtaining a finite set of vertex lists with no duplicates except ending where they start. None of this affects the integral $\int_P f(z)\,\mathrm{d}z$, which is now 0.

4. Cauchy's Theorem for Simple Curves

Let Ω be a region, let $f : \Omega \longrightarrow \mathbb{C}$ be differentiable, and let γ be a simple rectifiable closed curve in Ω whose interior lies in Ω. A *simple closed* curve is a loop with no self-intersections except that its endpoint is its start-point. A *rectifiable* curve is a curve of finite length. The seemingly self-evident fact that a simple closed curve has an interior and an exterior is the *Jordan Curve Theorem*, not at all trivial to prove. For example, the theorem fails for simple closed curves on a torus rather than in the plane, even though the plane and the torus are indistinguishable in the small; so the proof must make use of something quantifiable that distinguishes the plane from the torus.

Cauchy's Theorem for simple curves states that

$$\int_\gamma f(z)\,\mathrm{d}z = 0.$$

The proof requires a little topology. The first claim is that for some $\rho > 0$, the ρ-thickened version of the curve still lies in the region,

$$\bigcup_{z \in \gamma} \overline{B(z,\rho)} \subset \Omega. \tag{2}$$

Here $\overline{B(z,\rho)}$ is the *closed* ball about z of radius ρ. If Ω is all of $\mathbb{C}$ then the containment holds with $\rho = 1$. Otherwise, the complement Ω^c is nonempty, and so we can define the distance function from the curve to the complement

$$d : \gamma \longrightarrow \mathbb{R}^+, \qquad d(z) = \inf\{|z - w| : w \in \Omega^c\}.$$

This function is continuous, as follows. Let $\varepsilon > 0$ be given. Consider two points z, z' of γ such that $|z - z'| < \varepsilon/2$. There exists $w \in \Omega^c$ such that $|z - w| < d(z) + \varepsilon/2$. Compute,

$$|z' - w| = |z - w + z' - z| \leq |z - w| + |z' - z| < d(z) + \varepsilon/2 + \varepsilon/2 = d(z) + \varepsilon.$$

Thus $d(z') < d(z) + \varepsilon$, i.e., $d(z') - d(z) < \varepsilon$. Symmetrically, $d(z) - d(z') < \varepsilon$ as well, so $|d(z') - d(z)| < \varepsilon$ if $|z - z'| < \varepsilon/2$, showing that d is continuous as claimed. Also γ is compact, and so d has a minimum, which is positive. Denote this minimum 2ρ. Then $|z - w| \geq 2\rho$ for all $z \in \gamma$ and $w \in \Omega^c$, and the containment (2) follows.

Let R (for "ribbon") denote the thickened curve,

$$R = \bigcup_{z\in\gamma} \overline{B(z,\rho)} \subset \Omega.$$

Consider finitely many points $z_0, z_1, z_2, \dots, z_n = z_0$ of γ, in order of clockwise traversal, each within distance ρ of its predecessor along the length of γ. Consecutive points must then also be within distance ρ of each other in $\mathbb{C}$, and so the polygon P with the points as vertices lies in the ribbon R. If we add more points, this will not increase the distances between consecutive points along the curve, and so the resulting new polygon will still lie in R. (This is true even though adding more points, so that the distance along the curve between consecutive points is at most as big as before, can make the distance in $\mathbb{C}$ between consecutive points grow; for example, two points leading in and out of a hairpin turn are close, but the point at the turn is far from them both.)

Consider the sum

$$S = \sum_{j=1}^{n} f(z_j)(z_j - z_{j-1}).$$

This is a Riemann sum for the curve integral $\int_\gamma f(z)\,\mathrm{d}z$ that we want to equal zero, and by taking enough division points z_j we can make S as close to $\int_\gamma f(z)\,\mathrm{d}z$ as we wish.

Also, S is a Riemann sum for a polygon integral $\int_P f(z)\,\mathrm{d}z$, with P the polygon having vertices $z_0, \dots, z_n$, and this integral is zero.

However, the argument that adding more division points thus also makes S as close to zero as we wish isn't quite transparent. The problem is that while the curve γ is fixed in this discussion, so that adding more points z_j along γ refines Riemann sums for the *one particular* integral $\int_\gamma f(z)\,\mathrm{d}z$, adding those points also changes the polygon P and so the process is not refining sums for any particular polygon integral $\int_P f(z)\,\mathrm{d}z$. So even though polygon integrals are zero, a little more work is required to show that adding enough points makes S close to zero by making it close to a polygon integral.

Now let it be understood that with points $z_0, z_1, \dots, z_n = z_0$ chosen, P denotes their polygon. The difference between the relevant polygon integral and sum is

$$\int_P f(z)\,\mathrm{d}z - S = \sum_{j=1}^{n} \left(\int_{z_{j-1}}^{z_j} f(z)\,\mathrm{d}z - f(z_j)(z_j - z_{j-1}) \right),$$

with the integrals taken along the line segments $[z_{j-1}, z_j]$. This equality rewrites as

$$\int_P f(z)\,\mathrm{d}z - S = \sum_{j=1}^{n} \int_{z_{j-1}}^{z_j} (f(z) - f(z_j))\,\mathrm{d}z.$$

As explained above, the polygon P remains in the ribbon R as we add points. Also, the polygon P remains an inscribed polygon of γ as points are added, so that always

$$\operatorname{length}(P) \le \operatorname{length}(\gamma).$$

Because R is a compact subset of Ω, and because f is continuous on Ω, f is uniformly continuous on R. So, for any $\varepsilon > 0$, there exists some $\delta > 0$ such that for all $z, \tilde{z} \in R$,

$$|z - \tilde{z}| < \delta \implies |f(z) - f(\tilde{z})| < \varepsilon/\operatorname{length}(\gamma).$$

Finally, add enough points to make $|z_j - z_{j-1}| < \delta$ for all j. This puts everything in place for the final calculation,

$$\begin{aligned}
\left|\int_P f(z)\,\mathrm{d}z - S\right| &= \left|\sum_{j=1}^{n}\int_{z_{j-1}}^{z_j}(f(z)-f(z_j))\,\mathrm{d}z\right| \\
&\le \sum_{j=1}^{n}\left|\int_{z_{j-1}}^{z_j}(f(z)-f(z_j))\,\mathrm{d}z\right| \\
&\le \sum_{j=1}^{n}\int_{z_{j-1}}^{z_j}|f(z)-f(z_j)|\,|\mathrm{d}z| \\
&\le \sum_{j=1}^{n}(\varepsilon/\mathrm{length}(\gamma))\cdot|z_j - z_{j-1}| \\
&\le (\varepsilon/\mathrm{length}(\gamma))\cdot\mathrm{length}(P) \\
&\le \varepsilon.
\end{aligned}$$

Because the sum S is arbitrarily close to the curve-integral $\int_\gamma f(z)\,\mathrm{d}z$, and because it is arbitrary close to the polygon integral $\int_P f(z)\,\mathrm{d}z = 0$, Cauchy's Theorem for simple curves follows,

$$\int_\gamma f(z)\,\mathrm{d}z = 0.$$

A variant proof can be given if γ is piecewise $\mathcal{C}^1$ beyond being rectifiable, but it is not much different.