THREE BASIC IDEAS

The theory of complex-differentiable functions works out so nicely because of three ideas. This writeup previews them briefly, omitting technical details.

The first idea is that *because of Cauchy's Theorem, complex-differentiable functions have integral representation*. Let Ω be a region, let $f : \Omega \longrightarrow \mathbb{C}$ be differentiable, and let γ be a simple loop in Ω traversed counterclockwise. Assume that the interior of γ is entirely in Ω; that is, γ does not go around any holes of Ω. Then for any z inside γ, Cauchy's integral representation formula is

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - z}\,.$$

One could be led to the formula as follows. Let δ be a small circle about z. A short calculation shows that $\int_\delta \mathrm{d}\zeta/(\zeta - z) = 2\pi i$, so that (using the fact that δ is small, so that f on δ is close to $f(z)$) we can approximate

$$f(z) = f(z) \cdot \frac{1}{2\pi i} \int_\delta \frac{\mathrm{d}\zeta}{\zeta - z} \sim \frac{1}{2\pi i} \int_\delta \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - z}\,.$$

But for the last integral, Cauchy's Theorem allows us to deform δ to γ, suggesting the desired integral representation formula.

Turning these ideas into a proof is a matter of reversing the steps and quantifying the approximation. By Cauchy's Theorem we may integrate over the small circle δ instead. A previous writeup has shown that when the circle is small enough, the quotient $(f(\zeta) - f(z))/(\zeta - z)$ is bounded absolutely for all ζ on δ, because the quotient is the restriction of a continuous function on a closed ball about z. Here the crucial technical point is that the bound c is independent of the radius r of δ once the radius is small enough. So we have

$$\frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - z} = f(z)\frac{1}{2\pi i} \int_\delta \frac{\mathrm{d}\zeta}{\zeta - z} + \frac{1}{2\pi i} \int_\delta \frac{f(\zeta) - f(z)}{\zeta - z}\,\mathrm{d}\zeta.$$

The first term is $f(z)$, and the second term is bounded absolutely by rc. Because the left side is independent of r, which can be arbitrarily small, and because c doesn't change as r shrinks to 0, we have Cauchy's integral representation formula,

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - z}\,.$$

At first, the appeal of this formula may be hard to see. It seems to express the value of f at one point z in terms of the values of f at infinitely many points ζ, surely a uselessly inefficient thing to do. But because z is generic, in fact the formula shows that *the values of f at all points inside γ are determined by the values of f on γ*. This "filling in" property of complex-differentiable functions shows that they are somehow very rigid.

Even so, expressing a value of f as an integral involving values of f may still appear to be a needless complication. But at this level of mathematics, it is time to move beyond the calculus-class mindset that the integral sign is something to get rid of, and to realize that *representing functions as integrals lets us manipulate them and thus analyze them*. The next two points illustrate this.

The second idea is that *because complex-differentiable functions have integral representation, they have higher derivatives, which also have integral representation*. That is, if a function f has one complex derivative then it has infinitely many, and each derivative has an integral representation formula akin to the one for f itself. Indeed, the z-derivative of the ζ-integral representation of f passes through the integral sign to show that the higher derivatives exist by giving a formula for them,

$$\frac{f^{(n)}(z)}{n!} = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)\,\mathrm{d}\zeta}{(\zeta - z)^{n+1}}\,, \quad n = 0, 1, 2, \dots$$

Nothing remotely like this is true in real analysis, but here it is a consequence of the integral representation. Later writeups will justify passing the differentiation through the integral.

The third idea is that *because complex-differentiable functions and all of their derivatives have integral representation, complex-differentiable functions have power series representation*. Let γ now be a circle centered at some point a. For any z inside γ and any ζ on γ,

$$|z - a| < |\zeta - a|,$$

and so (using the geometric sum formula for the third equality to follow)

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - a) - (z - a)} = \frac{1}{(\zeta - a)\left(1 - \frac{z-a}{\zeta-a}\right)} = \sum_{n=0}^{\infty} \frac{(z-a)^n}{(\zeta - a)^{n+1}}\,.$$

Therefore, by the integral representation of f, by the geometric series just computed, by passing an infinite sum through an integral, and by the formula for higher derivatives, f is represented by its power series,

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - z} \\ &= \frac{1}{2\pi i}\int_\gamma f(\zeta)\sum_{n=0}^{\infty}\frac{(z-a)^n}{(\zeta-a)^{n+1}}\,\mathrm{d}\zeta \\ &= \sum_{n=0}^{\infty}\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)\,\mathrm{d}\zeta}{(\zeta-a)^{n+1}}(z-a)^n \\ &= \sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(z-a)^n. \end{aligned}$$

Later writeups will justify passing the infinite sum through the integral.

This quick overview shows that much of the study of complex-differentiable functions requires only sophomore calculus technique: path-integrals and power series. Also, it shows that some junior analysis will be required to justify the exchanges of limits involved in passing the derivative inside the integral and then passing the infinite sum back out. Finally, the imprecise reference earlier to the “holes” of the underlying region Ω on which f is defined hints that a correct handling of complex analysis needs to address topological issues as well.