# ISOGENY FROM SU(2) TO SO(3)

This writeup constructs a 2-to-1 epimorphism $\mathrm{SU}(2) \longrightarrow \mathrm{SO}(3)$, quickly demonstrating methods by example without full discussion. In general, a group that doubly covers an orthogonal group is called a *spin group*. See Paul Garrett's writeup

`http://www-users.math.umn.edu/~garrett/m/v/sporadic_isogenies.pdf`

for many more examples.

## 1. Unitary group and its Lie algebra

For the unitary group $\mathrm{U}(2) \subset \mathrm{GL}_2(\mathbb{C})$, having Lie algebra $\mathfrak{u}(2) \subset \mathrm{M}_2(\mathbb{C})$, the condition

$$1 = (\overline{e^{tx}})^{\mathsf{T}} e^{tx} = e^{t\overline{x}^{\mathsf{T}}} e^{tx} \quad \text{for } x \in \mathfrak{u}(2)$$

differentiates at 0 to $0 = \overline{x}^{\mathsf{T}} + x$; and conversely if $\overline{x}^{\mathsf{T}} = -x$ then

$$(\overline{e^{tx}})^{\mathsf{T}} e^{tx} = e^{t\overline{x}^{\mathsf{T}}} e^{tx} = e^{-tx} e^{tx} = 1.$$

Thus the Lie algebra consists of the skew hermitian matrices. Here $\mathrm{U}(2)$ and $\mathfrak{u}(2)$ are a real Lie group and Lie algebra notwithstanding their complex entries. Their shared real dimension is 4. The Lie algebra $\mathfrak{su}(2)$ of the special unitary group $\mathrm{SU}(2)$ carries the additional condition that the trace vanishes,

$$\mathfrak{su}(2) = \{ x \in \mathrm{M}_2(\mathbb{C}) : \overline{x}^{\mathsf{T}} = -x,\ \operatorname{tr} x = 0 \}.$$

This reduces its dimension to 3, also the manifold dimension of $\mathrm{SU}(2)$. Here the argument is that the condition $\det e^{tx} = 1$ is $e^{t \operatorname{tr} x} = 1$, which differentiates at $t = 0$ to $\operatorname{tr} x = 0$; and conversely if $\operatorname{tr} x = 0$ then $\det e^{tx} = e^{t \operatorname{tr} x} = e^0 = 1$.

The $\mathfrak{su}(2)$ conditions $\overline{x}^{\mathsf{T}} = -x$ and $\operatorname{tr} x = 0$ are preserved under addition, real scaling, and the Lie bracket. For example,

$$(\overline{rx})^{\mathsf{T}} = \overline{r}\,\overline{x}^{\mathsf{T}} = -r\,x \quad \text{for real } r,$$

and

$$(\overline{xy - yx})^{\mathsf{T}} = \overline{y}^{\mathsf{T}} \overline{x}^{\mathsf{T}} - \overline{x}^{\mathsf{T}} \overline{y}^{\mathsf{T}} = (-y)(-x) - (-x)(-y) = yx - xy = -(xy - yx).$$

## 2. Inner Product, Invariance

A real symmetric bilinear inner product on $\mathfrak{su}(2)$ is

$$\langle \cdot, \cdot \rangle : \mathfrak{su}(2) \times \mathfrak{su}(2) \longrightarrow \mathbb{R}, \qquad \langle x, y \rangle = \mathrm{Re}(\operatorname{tr}(xy)).$$

The group $\mathrm{SU}(2)$ acts on the algebra $\mathfrak{su}(2)$ by conjugation,

$$g \cdot x = gxg^{-1},$$

and this action preserves the inner product,

$$\langle g \cdot x, g \cdot y \rangle = \langle x, y \rangle.$$

Indeed, to see that $g \cdot x$ again lies in $\mathfrak{su}(2)$ for all $g \in \mathrm{SU}(2)$ and $x \in \mathfrak{su}(2)$, note that because $g$ and $e^{\mathbb{R}x}$ and $g^{-1}$ lie in $\mathrm{SU}(2)$, also

$$e^{\mathbb{R}gxg^{-1}} = ge^{\mathbb{R}x}g^{-1} \quad \text{lies in } \mathrm{SU}(2),$$

and to see that the action preserves the inner product, compute that

$$\langle g\cdot x, g\cdot y\rangle = \mathrm{Re}(\mathrm{tr}\,(gxg^{-1}gyg^{-1})) = \mathrm{Re}(\mathrm{tr}\,(xy)) = \langle x, y\rangle.$$

Note that the results in this section rely only on general Lie group and Lie algebra properties, not on any particulars of the specific Lie group SU(2) and its Lie algebra $\mathfrak{su}(2)$.

## 3. Orthogonal Basis, Group Action, Isogeny

The $\mathfrak{su}(2)$-basis

$$x_1 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \qquad x_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad x_3 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

is orthogonal under the inner product, with $\langle x_i, x_i\rangle = -2$ for $i = 1, 2, 3$. For example,

$$\langle x_1, x_1\rangle = \mathrm{Re}(\mathrm{tr}\,(\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix})) = -2, \qquad \langle x_2, x_3\rangle = \mathrm{Re}(\mathrm{tr}\,(\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix})) = 0.$$

Thus we have the $3\times 3$ matrix

$$[\langle x_i, x_j\rangle] = -2I_3.$$

The action of any group element

$$g = \begin{bmatrix} a & b \\ -\overline{b} & \overline{a} \end{bmatrix} = \begin{bmatrix} \alpha_1 + i\alpha_2 & \beta_1 + i\beta_2 \\ -\beta_1 + i\beta_2 & \alpha_1 - i\alpha_2 \end{bmatrix} \in \mathrm{SU}(2)$$

on the basis elements is a matter of direct computation, albeit a bit tedious,

$$\begin{aligned} g\cdot x_1 &= (\alpha_1^2 + \alpha_2^2 - \beta_1^2 - \beta_2^2)x_1 + 2(\alpha_2\beta_1 + \alpha_1\beta_2)x_2 + 2(\alpha_2\beta_2 - \alpha_1\beta_1)x_3 \\ g\cdot x_2 &= 2(\alpha_2\beta_1 - \alpha_1\beta_2)x_1 + (\alpha_1^2 - \alpha_2^2 + \beta_1^2 - \beta_2^2)x_2 + 2(\alpha_1\alpha_2 + \beta_1\beta_2)x_3 \\ g\cdot x_3 &= 2(\alpha_1\beta_1 + \alpha_2\beta_2)x_1 + 2(-\alpha_1\alpha_2 + \beta_1\beta_2)x_2 + (\alpha_1^2 - \alpha_2^2 - \beta_1^2 + \beta_2^2)x_3. \end{aligned}$$

This shows that the map from SU(2) to the special orthogonal group SO(3) is the quadratic map

$$\varphi : g \longmapsto \begin{bmatrix} \alpha_1^2 + \alpha_2^2 - \beta_1^2 - \beta_2^2 & 2(\alpha_2\beta_1 - \alpha_1\beta_2) & 2(\alpha_1\beta_1 + \alpha_2\beta_2) \\ 2(\alpha_2\beta_1 + \alpha_1\beta_2) & \alpha_1^2 - \alpha_2^2 + \beta_1^2 - \beta_2^2 & 2(-\alpha_1\alpha_2 + \beta_1\beta_2) \\ 2(-\alpha_1\beta_1 + \alpha_2\beta_2) & 2(\alpha_1\alpha_2 + \beta_1\beta_2) & \alpha_1^2 - \alpha_2^2 - \beta_1^2 + \beta_2^2 \end{bmatrix}.$$

Let the matrix in the previous display be denoted $A_g$. To argue that $A_g$ is orthogonal, introduce the map that converts elements of $\mathfrak{su}(2)$ into $\mathbb{R}^3$-vectors,

$$v : \mathfrak{su}(2) \xrightarrow{\sim} \mathbb{R}^3, \quad v(\sum_{i=1}^{3} c_i x_i) = \sum_{i=1}^{3} c_i e_i, \quad (e_i \text{ the standard basis vectors}).$$

Because $A_g$ is the matrix of of the $g$-action, we have a commutative diagram

$$\begin{array}{ccc} \mathfrak{su}(2) & \xrightarrow{g} & \mathfrak{su}(2) \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle v} \\ \mathbb{R}^3 & \xrightarrow{A_g} & \mathbb{R}^3 \end{array}$$

which is to say,

$$A_g v(x) = v(g\cdot x), \quad g \in \mathrm{SU}(2),\ x \in \mathfrak{su}(2). \tag{1}$$

And note that by our inner product calculations on the $\mathfrak{su}(2)$ basis elements,

$$\langle x, x' \rangle_{\mathfrak{su}(2)} = -2\langle v(x), v(x') \rangle_{\mathbb{R}^3}, \quad x, x' \in \mathfrak{su}(2). \tag{2}$$

Now compute for any $g \in \mathrm{SU}(2)$ and any $x, x' \in \mathfrak{su}(2)$, recalling for the third equality that the SU(2) action preserves the $\mathfrak{su}(2)$ inner product,

$$\begin{aligned} \langle A_g v(x), A_g v(x') \rangle_{\mathbb{R}^3} &= \langle v(g \cdot x), v(g \cdot x') \rangle_{\mathbb{R}^3} && \text{by (1)} \\ &= (-1/2)\langle g \cdot x, g \cdot x' \rangle_{\mathfrak{su}(2)} && \text{by (2)} \\ &= (-1/2)\langle x, x' \rangle_{\mathfrak{su}(2)} && \text{as just recalled} \\ &= \langle v(x), v(x') \rangle_{\mathbb{R}^3} && \text{by (1) again.} \end{aligned}$$

This shows that $A_g$ is orthogonal.

The map $g \mapsto A_g$ is innately a homomorphism, because the action property $(gg')\cdot x = g\cdot(g'\cdot x)$ for $g, g' \in \mathrm{SU}(2)$ and $x \in \mathfrak{su}(2)$ combines with the fact that matrix multiplication is compatible with linear map composition to give $A_{gg'} = A_g A_{g'}$.

To determine the kernel of the map $g \mapsto A_g$, note that the diagonal conditions

$$\alpha_1^2 + \alpha_2^2 - \beta_1^2 - \beta_2^2 = \alpha_1^2 - \alpha_2^2 + \beta_1^2 - \beta_2^2 = \alpha_1^2 - \alpha_2^2 - \beta_1^2 + \beta_2^2 = 1$$

are $\alpha_1^2 - 1 = \alpha_2^2 = \beta_1^2 = \beta_2^2$. Now the $(1,2)$-entry condition $\alpha_2\beta_1 - \alpha_1\beta_2 = 0$ is $\pm\beta_2^2 = \pm\sqrt{\beta_2^2 + 1}\beta_2$; if $\beta_2 \neq 0$ then canceling $\beta_2$ and then squaring both sides gives the impossible condition $\beta_2^2 = \beta_2^2 + 1$, so $\beta_2 = 0$. This forces $g = \pm 1_2$. Conversely, if $g = \pm 1$ then $A_g = 1$. In sum, the map has kernel is $\pm 1_2$.

Because the manifold dimension 3 of O(3) matches that of the connected group SU(2), the map $g \mapsto A_g$ surjects to the connected component SO(3) of its codomain.