

**CALCULUS AND ANALYSIS IN EUCLIDEAN SPACE:
ADDITIONS AND CORRECTIONS
FIRST PRINTING**

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As of autumn 2019, the second printing should be available. A separate list of additions and corrections will be maintained for it. To check whether a copy of the book is the second printing, see if it has the first few additions and corrections here.

Preface

- Page ix: Remove “, too many for a short list here to do them justice” near the bottom of the page.

Chapter 1

- Pages 2–3: Near the bottom of page 2, remove “A version of completeness. . . and therefore so is completeness.” Change the first word of the next sentence from “Another” to “One”. At the top of page 3, change “Both statements of completeness are existence statements” to “This statement of completeness is an existence statement”. Near the bottom of page 3, change “the assumption that \mathbb{R} is complete in the sense of binary search sequences or in the sense of set-bounds. A third. . .” to “the assumption that \mathbb{R} satisfies the set-bound criterion for completeness. A second. . .”. After Theorem 1.1.6, change “either of the other two” to “the first one”, and change “the other two unless” to “the first one unless”. (All of these changes are because completeness as a binary search criterion is apparently not well known.)
- Page 9: On line 2, change “First,” to “We interpret the empty nested integral $I_0(x)$ to be identically 1. Next,”. Halfway down the page, in the display after “and the answer in general is”, change “ $k \in \mathbb{Z}^+$ ” to “ $k = 0, 1, 2, \dots$ ”. Three lines below that, change “ $k = 1, 2, 3, \dots$ ” to “ $k = 0, 1, 2, \dots$ ”.
- Page 16: On line 7, “polynomial growth” should be “exponential growth”.

Chapter 3

- Page 82: In Lemma 3.3.3 part (1), remove “. Specifically”. In part (3), remove “specifically”.
- Page 90: Replace the last two lines by, “For one consequence of the determinant’s existence, with no reference to its uniqueness, consider the standard basis of \mathbb{R}^n taken in order,”
- Page 91: On line 4, change “negates” to “changes the sign of”. On lines 7, change “Since det” to “Since the determinant”. On line 16, change “exchanges” to “pair-exchanges”. On line 21, change “skewness” to “skew-symmetry”. On line 22, change “See” to “Alternatively, see”. On line 24, remove “ clearly, with reference to the exercise”. Replace the following displayed text and then the rest of the paragraph by

“The existence of a determinant with no reference to its uniqueness, or an argument that makes no reference to the determinant

at all, shows that every rearrangement of n objects has a well-defined parity, meaning that either all sequences of pair-exchanges that put the objects back in order have even length or all such sequences have odd length.

In the next section we will show that there are as many candidate determinants (multilinear skew-symmetric normalized functions) as there are ways to assign a parity to each rearrangement of n objects, with no assumption that any determinant exists. So there could be as many as $2^{n!}$ candidate determinants, in the extreme case that each rearrangement can be put back in order by an odd number of pair-exchanges and by an even number. And in the next section we will use one particular assignment of a parity to each rearrangement to show that a determinant exists. As in the previous displayed text, once a determinant exists, only one parity-assignment function exists, and so the determinant is unique. The logic here is subtle, and so the reader may prefer to rely on Exercise 3.5.2 to defray any concern about arguing in a circle. If the uniqueness of parity is established first then the ideas lay themselves out more clearly: a unique candidate determinant presents itself, and we show that it works.” Replace the next sentence by, “The next result is the crucial property of the determinant, in consequence of its characterizing properties.” In Theorem 3.5.2, before the display, add, “the determinant of the matrix product is the product of the scalar determinants,”, and after the display, change “In particular” to “Further”.

- Page 95: At the end of the section, before the exercises, add, “We will give a more efficient determinant algorithm in the next section.”
- Page 100: On line 13, change “determinant formula” to “determinant formulas”.
- Page 101: Replace the four lines before Definition 3.6.2 by “Because $(-1)^\pi$ arises from a specific method of undoing any permutation—exchange out-of-order neighboring pairs until none remain—it conceivably need not be the only parity function of permutations. Further, the argument here has shown that for any parity function sgn of permutations, the function

$$\det_{sgn}(r_1, r_2, \dots, r_n) = \sum_{\pi=(i,j,\dots,p)} sgn(\pi) a_{1i} a_{2j} \dots a_{np}$$

is a possible formula for a multilinear skew-symmetric normalized function, and these are the only candidates. As discussed in the previous section, either we already know that $(-1)^\pi$ is the unique parity of each permutation π by Exercise 3.5.2, or we will know it as soon as the function constructed with it in the penultimate display is shown to be multilinear, skew-symmetric, and normalized.”

- Page 103: Starting at line 3, replace “. . . properties. We don’t yet know. . .” to the end of the paragraph with “. . . properties, its uniqueness dependent on its doing so if we haven’t already shown that each permutation has a unique sign. That is, we have now proved the uniqueness but not yet the existence of the determinant in Theorem 3.5.1, the uniqueness possibly provisional on the existence.”

- Page 104: Replace the two paragraphs starting “So a unique” and “The reader” with “So a determinant function with the stipulated behavior exists, making our $(-1)^\pi$ the only possible sign of each permutation π if we don’t know this already, and thus showing that the determinant is unique. Another construction of a determinant function, with no reference to permutations at all but proceeding instead by induction on the dimension n of the matrices, is given in Exercise 3.6.12. And we have seen that every multilinear skew-symmetric function must be a scalar multiple of the determinant. The last comment necessary to complete the proof of Theorem 3.5.1 is that since the determinant is multilinear and skew-symmetric, so are its scalar multiples. This fact was shown in Exercise 3.5.3.

The reader is invited to contemplate how unpleasant it would have been to prove the various theorems about the determinant in the previous section using the unwieldy determinant formula, with its $n!$ terms, each an n -fold product. That said, the theorems really can be shown directly from the formula. For example, to prove that $\det(A^\top) = \det(A)$, one can write

$$\det(A^\top) = \sum_{\pi \in S_n} (-1)^\pi a_{\pi(1)1} a_{\pi(2)2} \cdots a_{\pi(n)n},$$

and then persuade oneself that this is also the sum over the permutations π' that undo the permutations π , and the undo-permutations have the same signs as the originals,

$$\det(A^\top) = \sum_{\pi' \in S_n} (-1)^{\pi'} a_{1\pi'(1)} a_{2\pi'(2)} \cdots a_{n\pi'(n)},$$

and this is $\det(A)$. Here we are adumbrating basic ideas from *group theory*.”

- Page 108: Add Exercise 3.6.12 as follows: “This exercise constructs a determinant with no reference to permutations or their signs, inductively on the dimension n of the matrix. Define $\det_1([a]) = a$, and then

$$\det_n(A) = \sum_{j=1}^n (-1)^{1+j} a_{1j} \det_{n-1}(A^{1j}), \quad n \geq 2,$$

where A^{1j} is the $(n-1) \times (n-1)$ matrix obtained by removing the first row and j th column of A . One can start instead with $\det_0([\]) = 1$ and then the displayed formula for $n \geq 1$. Show by induction on n that \det_n is multilinear, alternating (hence skew-symmetric), and normalized as a function of the rows of A .”

- Page 114: The discussion can be improved, e.g., “As for boxes, scaling any spanning vector of a parallelepiped by a real number a magnifies the volume by $|a|$, and so we have

$$\text{vol } T\mathcal{B}' = \text{vol } T\mathcal{B} \cdot a_1 \cdots a_n.$$

But also,

$$a_1 \cdots a_n = \text{vol } \mathcal{B}'.$$

That is, the volume of the T -image of any box is a constant multiple of the volume of the box, regardless of the box’s location or side lengths, the constant being the volume of $T\mathcal{B}$, the T -image of the unit box \mathcal{B} . Call this constant magnification factor t . Thus,

$$\text{vol } T\mathcal{B}' = t \cdot \text{vol } \mathcal{B}' \quad \text{for all boxes } \mathcal{B}'.$$

- Page 126: On line 14, change “Since we need” to “Since we need the cross product to have components”, and on line 18, change “is” to “is to construct the cross product from these components,”.
- Page 129: In Exercise 3.10.10, change “conclude if” to “conclude if instead”.

Chapter 4

- Page 134: In the first and third bullets of Proposition 4.2.2, the “little oh” symbol should be o .
- Page 135: The second display should say “ $\psi(h) = h_i$ ”, not “ $\varphi(h) = h_i$ ”.
- Page 138: The third-to-last display should say “for every $d > 0$ ” rather than “for any $d > 0$ ”.
- Page 145: Proposition 4.3.4 should be titled “Differentiability implies continuity”.
- Page 147: Proposition 4.4.2 should be titled “Linearity of differentiation”.
- Page 149: Replace $ak + bh$ by $bh + ak$ in statement (1) of Lemma 4.4.4, and replace $-ak - bh$ by $-bh - ak$ twice in its proof.
- Page 149: On the last line, replace the equality with

$$D(fg)_a(h) = g(a)Df_a(h) + f(a)Dg_a(h).$$

- Page 150: Replace the first display with

$$D\left(\frac{f}{g}\right)_a(h) = \frac{g(a)Df_a(h) - f(a)Dg_a(h)}{g(a)^2}.$$

End the first line of the third display with a period, and remove the second line of that display. Immediately after the third display, remove “, since h is arbitrary”.

- Page 152: A better version of Exercise 4.4.8(a) is: “Show that if f is multilinear and $a_1, \dots, a_k, h_1, \dots, h_k \in \mathbb{R}^n$ then for any $j \in \{2, \dots, k\}$, $f(h_1, \dots, h_j, a_{j+1}, \dots, a_k)$ is $o(h_1, \dots, h_k)$. The same result holds if any j inputs to f are h ’s, rather than the first j inputs, because permuting the inputs of a multilinear function creates another multilinear function. Flesh this argument out as much as feels necessary for your understanding.”
- Page 155, line 9: Change “ $f_i a$ ” to “ $f_i(a)$ ”.
- Page 163: Remove the box around -2 .
- Page 163: Add a new paragraph at the end of the section, as follows: “For another example, the function $f(x) = x^x$ is usually differentiated as follows in one-variable calculus: Consider the related function $\ln(f(x)) = \ln(x^x) = x \ln(x)$, and take derivatives of both sides to get $f'(x)/f(x) = 1 + \ln(x)$; thus $f'(x) = x^x(1 + \ln(x))$. On the other hand, if we differentiate x^x treating the first x as variable and the second x as constant then we get $x x^{x-1} = x^x$, and if we differentiate x^x treating the first x as constant and the second x as variable then we get $x^x \ln(x)$; the sum of these two sort-of-derivatives is $x^x(1 + \ln(x))$, the derivative of x^x as computed a moment ago. The method of treating the two x ’s as independent has produced the right answer, despite its illegality. This can’t be a coincidence, and it isn’t. In general, if $F(x_1, \dots, x_n)$ is a differentiable function of many variables then the derivative of the one-variable function $f(x) = F(x, x, \dots, x)$ is $f'(x) = \sum_{i=1}^n D_i F(x, x, \dots, x)$. Exercise 4.5.10 is to prove this formula as an immediate consequence of the chain rule, and then to use it to establish a result known as Leibniz’s Rule. Exercise 4.5.11(a) is to use this formula to

differentiate the function $f(x) = x^{x^x}$, and more generally Exercise 4.5.11(b) is to differentiate the function $f(x) = x^{x^{x^x}}$.

- Page 164: Add a part (a) to Exercise 4.5.10 as follows: “(a) Consider a function $f(x) = F(x, x, \dots, x)$ where $F(x_1, x_2, \dots, x_n)$ is a differentiable function of n variables. Note that $f = F \circ \gamma$ where $\gamma(x) = (x, x, \dots, x)$, and use this to show that $f'(x) = \sum_{i=1}^n D_i F(x, x, \dots, x)$.” Change the beginning of the current version of the exercise from “Let. . .” to “(b) (Leibniz’s Rule.) Let. . .”. On the next page, after the five-line display and the comment in parentheses, change the end of the exercise to “Let $\alpha, \beta : \mathbb{R} \rightarrow \mathbb{R}$ be differentiable functions. Define a function

$$G : \mathbb{R} \rightarrow \mathbb{R}, \quad G(x) = \int_{y=\alpha(x)}^{\beta(x)} f(x, y) dy.$$

Thus x affects G in three ways: as a contributor the lower and upper limits of integration, and as a parameter for the integrand. What is $dG(x)/dx$? (Hint: $G(x) = H(x, x, x)$ where $H(x_1, x_2, x_3) = \int_{y=\alpha(x_1)}^{\beta(x_2)} f(x_3, y) dy$.)”

- Page 172: Change the comment before Exercise 4.6.2 to “For the rest of these exercises, assume that the relevant functions are \mathcal{C}^2 .”
- Page 176: Replace the “Irkisomely” paragraph with “The eminently plausible formula $f'' = (f')'$ indeed holds, provided that we view f' as a mapping to \mathbb{R}^n , each of whose outputs is an ordered list with no shape rather than a row vector. Thus

$$(f')'(a) = f''(a) \quad \text{for interior points } a \text{ of } A.$$

- Page 179: About halfway down the page, replace the two lines beginning “The previous display” with “The previous display can be rephrased as $\varphi'(t) = \langle f'(\gamma(t)), h \rangle$, and so the chain rule and the symmetry of f'' give”
- Page 182: Change “ G ” to “ T ” in the first line after the second display.
- Page 188: The left side of Figure 4.12 has different horizontal and vertical scales; scaled correctly, it would show the two vectors at right angles.
- Page 196: In Exercise 4.8.8(a), change “Solve” to “Similarly to an example in the section, solve”.

Chapter 5

- Page 206: Replace the paragraph after the third display with “This formula combines with Proposition 4.3.4 (differentiability implies continuity) and Corollary 3.7.3 (the entries of the inverse matrix are continuous functions of the entries of the matrix) to show that since the mapping is continuously differentiable and the local inverse is differentiable, the local inverse is continuously differentiable: If y varies slightly, then so does x because f^{-1} is continuous, hence so does $f'(x)$ because f' is continuous, hence so does $f'(x)^{-1}$, which is $(f^{-1})'(y)$. Thus we need to show only that the local inverse exists and is differentiable.”
- Page 208: In (5.5) replace “ $y - f(a)$ ” and “ $f(x) - y$ ” with “ $f(a) - y$ ” and “ $f(x) - y$ ”. Change the next sentence to “That is, for every point y of W , $f(a)$ is closer to y than every point of $f(\partial B)$ is close to y .” At the bottom of the page, change “from $f(x)$ to y ” to “between $f(x)$ and y ”, and in the last display change “ $y - f(x)$ ” and “ $y_i - f_i(x)$ ” to “ $f(x) - y$ ” and “ $f_i(x) - y$ ”.

- Page 209: In the first display, change “ $= -2$ ” and “ $y_i - f_i(x)$ ” to “ $= 2$ ” and “ $f_i(x) - y_i$ ”. In the second display, change “ $y_1 - f_1(x)$ ” and “ $y_n - f_n(x)$ ” to “ $f_1(x) - y_1$ ” and “ $f_n(x) - y_n$ ”. In the third display, change “ $y - f(x)$ ” to “ $f(x) - y$ ”, and make the same change in the second line of text after the third display. In the third line of text after the third display, change “ $y = f(x)$ ” to “ $f(x) = y$ ”.
- Page 239: Change the last constraint in Exercise 5.4.5 to $yz + x = 4$.

Chapter 6

- Page 268: Exercise 6.2.7 should say that upper sums for f can be bigger than *lower* sums for g .
- Page 271: Replace the “To prove instead” paragraph with “To prove instead that the function $f : \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = x^2$ is sequentially continuous on \mathbb{R} , again take any $x \in \mathbb{R}$. Consider any sequence $\{x_\nu\}$ in \mathbb{R} converging to x . To show that the sequence $\{f(x_\nu)\}$ in \mathbb{R} converges to $f(x)$, compute that by sequence limit properties,

$$\{f(x_\nu)\} = \{x_\nu^2\} \xrightarrow{\nu} x^2 = f(x).$$

Since x is arbitrary, f is sequentially continuous on \mathbb{R} . Note how much easier this is than the ϵ - δ argument. Sequential continuity can be easier to establish, as shown here, but also it can be harder to exploit.”

- Page 280: A tidier proof of Theorem 6.4.3 is, “Define $F_2 : [a, b] \rightarrow \mathbb{R}$ by $F_2(x) = \int_a^x F'$. Then $F_2' = F'$ by the preceding theorem, so (Exercise 6.4.3) there exists a constant c such that $F_2(x) = F(x) + c$ for all $x \in [a, b]$. Thus, because $F_2(a) = 0$,

$$\int_a^b F' = F_2(b) - F_2(a) = F(b) + c - F(a) - c = F(b) - F(a).”$$

- Page 282: Replace the paragraph after Corollary 6.4.4 by, “The formula in the corollary is the formula for **integration by inverse substitution**. To obtain it from (6.4), substitute $f \cdot (\phi^{-1})'$ for f in (6.4) to get

$$\int_a^b (f \circ \phi) \cdot ((\phi^{-1})' \circ \phi) \cdot \phi' = \int_{\phi(a)}^{\phi(b)} f \cdot (\phi^{-1})'.$$

By the chain rule, $((\phi^{-1})' \circ \phi) \cdot \phi' = (\phi^{-1} \circ \phi)' = 1$, and so we have the result.”

- Page 295: In the statement of Fubini’s theorem, expand the “Suppose . . .” sentence to, “Suppose that for each $x \in [a, b]$, the cross-sectional integral $\int_{y=c}^d f(x, y)$ exists; this happens if the cross-sectional function $\varphi_x : [c, d] \rightarrow \mathbb{R}$ given by $\varphi_x(y) = f(x, y)$ is continuous as a function of y except on a subset of length zero, and in particular this happens if S contains only finitely many points (possibly none) having first coordinate x .” This expanded statement covers cases such as $f : [0, 2] \times [0, 1] \rightarrow \mathbb{R}$ where $f(x, y) = 0$ if $x \leq 1$ and $f(x, y) = 1$ if $x > 1$, with the constant function $\varphi_1 = 0$ integrable along the cross-section at $x = 1$ even though this cross-section is the discontinuity set S of f .
- Page 305: After the four-line display, replace the two lines of text with “We show that $\int_{y=c}^d D_1 f(t, y)$ is a continuous function of t . Fix t , and let $\epsilon > 0$ be given. The continuity of $D_1 f$ on its compact domain $[a, b] \times [c, d]$ is uniform, so for some $\delta > 0$, for all \tilde{t} such that $|\tilde{t} - t| < \delta$, we have $|D_1 f(\tilde{t}, y) - D_1 f(t, y)| <$

$\varepsilon/(d-c)$ for all $y \in [c, d]$. Thus for all such \tilde{t} ,

$$\left| \int_{y=c}^d D_1 f(\tilde{t}, y) - \int_{y=c}^d D_1 f(t, y) \right| \leq \int_{y=c}^d |D_1 f(\tilde{t}, y) - D_1 f(t, y)| < \varepsilon.$$

This proves the claimed continuity. Now Theorem 6.4.1 says that the derivative of the iterated integral is the inner integral evaluated at $t = x$,” Also, end the next display with a period rather than a comma, and change the next line of text to “This is the desired result.”

- Page 306: Change the first sentence of Exercise 6.6.13(a) to “Sketch $S_n(r)$ for $n = 1, 2, 3$, with your sketches for $n = 2$ and $n = 3$ showing that $S_n(r)$ is a disjoint union of cross-sectional $(n-1)$ -dimensional simplices of side $r - x_n$ at height x_n as x_n varies from 0 to r .”
- Page 317: Replace the “Thus” before the second display with “The absolute determinant of the derivative matrix of Φ is the obvious volume-dilation constant,”. Immediately after that, change $f(x, y, z)$ from Cz^2 to z^2 . In the third display, write the constant in front of the last integral as $abc \cdot c^2$. In the fourth integral, eliminate the constant in front of the integrals, so that the end result is just $\frac{4\pi}{15}$.
- Page 322: Add a new Exercise 6.7.15 as follows. “This exercise heuristically derives *Stirling’s formula*,

$$n! \sim \sqrt{2\pi n} (n/e)^n, \quad n \gg 0.$$

(a) Show that because $n! = \Gamma(n+1)$, it follows that $n! = \int_{t=0}^{\infty} e^{n \ln t - t} dt$.

(b) With n fixed and t variable, show that the quantity $n \ln t - t$ takes its maximum value $n \ln n - n$ at $t = n$, where its first derivative is 0 and its second derivative is $-1/n$. Thus the quantity’s quadratic approximation about its maximizing point is $n \ln n - n - \frac{1}{2n}(t-n)^2$.

(c) In the integral expression of $n!$ from (a), replace $n \ln t - t$ by its quadratic approximation from (b) to get

$$\Gamma(n+1) \sim (n/e)^n \int_{t=0}^{\infty} e^{-\frac{1}{2n}(t-n)^2} dt.$$

The quantity $t - n$ runs through $(-n, \infty)$ as t runs through $(0, \infty)$. Thus, assuming that $n \gg 0$, replace $t - n$ by t and extend the integration to all of \mathbb{R} to get

$$\Gamma(n+1) \sim (n/e)^n \int_{t=-\infty}^{\infty} e^{-\frac{1}{2n}t^2} dt.$$

Replace t by $\sqrt{2n}t$, and use exercise 6.7.13 to evaluate the resulting integral and obtain Stirling’s formula.

(d) Use various exercises from this section to show that the centroid of the upper half n -ball has last coordinate

$$\bar{x}_n = \frac{2^{n+1}}{\pi(n+1)} \cdot \frac{(n/2)!^2}{n!},$$

and then use Stirling’s formula to show that asymptotically, as n grows,

$$\bar{x}_n \sim \sqrt{\frac{2}{\pi n}}.$$

Thus the n -ball is concentrated ever more toward its center.”

Increment the numbers of the remaining section 6.7 exercises.

- Pages 322–323: In the old Exercise 6.7.15, add at the end of part (a) “Thus

$$\alpha^{-s} = \frac{1}{\Gamma(s)} \int_{t=0}^{\infty} t^s e^{-\alpha t} \frac{dt}{t}, \quad \alpha > 0.”$$

In part (c), the displayed formula in Exercise 6.7.15(c) should have e^{-t} where it has e^{-s} . In the old Exercise 6.7.16, replace the first sentence of part (a) with “Recall from Exercise 6.7.15(a) that $\alpha^{-s} = \frac{1}{\Gamma(s)} \int_{t=0}^{\infty} t^s e^{-\alpha t} \frac{dt}{t}$ for all $\alpha > 0$.” But this correction should then be altered when the 6.7 exercises are renumbered.

- Section 6.8: This section can be omitted if we strengthen the hypotheses of the change of variable theorem to assume that $\Phi(K)$ has boundary of volume zero. For all 2-dimensional and 3-dimensional examples in a calculus class, one can see this to be the case.

Chapter 8

- Page 379, line –6: Exchange “lighter” and “darker”.

Chapter 9

- Page 428: Near the bottom of the page, replace the “The integral can be rewritten...” sentence with “The integral can be rewritten as follows:

$$\text{For curves } \gamma = (x, y, z) : [a, b] \longrightarrow \mathbb{R}^3, \quad \int_{\gamma} dx = x(b) - x(a).”$$

- Page 448: In the “...(now dropping the wedges from the notation)...” sentence, drop the wedge from the notation $dx \wedge dy$.
- Page 451: Add a thin space before “ $dx_{(1,\dots,n)}$ ” in the first display in the proof.
- Page 459: A better end to the proof of Theorem 9.10.1 is

$$\begin{aligned} \int_{\Phi} f dx_I &= \int_D (f \circ \Phi) \det \Phi'_I && \text{by definition, as in (9.14)} \\ &= \int_{\Delta^D} (f \circ \Phi) \det \Phi'_I du_1 \wedge \dots \wedge du_k && \text{by Exercise 9.5.4} \\ &= \int_{\Delta^D} \Phi^* f \cdot \Phi^* dx_I && \text{by Theorem 9.9.3} \\ &= \int_{\Delta^D} \Phi^*(f dx_I) && \text{by Theorem 9.9.4(2).} \end{aligned}$$

- Page 466: In Exercise 9.11.2, insert part (a): “Let $\omega = f(x, y) dx \wedge dy$ be a form on \mathbb{R}^2 , so that $d\omega = 0$. Find and confirm an antiderivative of ω .” Make the current exercise part (b).
- Page 472: On the second-to-last line, after “pairwise.” add “(See Exercise 9.13.8.)”
- Page 474: $\Delta_{1,b}^3$ should be $\Delta_{3,b}^3$ in two places toward the bottom of the page.
- Page 476: In Exercise 9.13.7, change the first two components of Φ to $(b + at \cos v) \cos u$ and $(b + at \cos v) \sin u$.
- Page 476: As the text explains, there is no reason to compute that the double boundary operator is zero, but nonetheless, add Exercise 9.13.8 to do so as follows. “This exercise gives a self-contained proof that the double boundary operator is identically zero. It suffices to show this for the double boundary on the standard k -cube, where $k \geq 2$.

(a) Explain why the double boundary is

$$\partial^2 \Delta^k = \sum_{i=1}^k \sum_{\alpha=0}^1 \sum_{j=1}^{k-1} \sum_{\beta=0}^1 (-1)^{i+j+\alpha+\beta} \Delta_{i,\alpha}^k \circ \Delta_{j,\beta}^{k-1}.$$

(b) Show that if $i \leq j$ then we have

$$\Delta_{i,\alpha}^k \circ \Delta_{j,\beta}^{k-1}(u_1, \dots, u_{k-2}) = (u_1, \dots, u_{i-1}, \alpha, u_i, \dots, u_{j-1}, \beta, u_j, \dots, u_{k-2}),$$

with α in the i th slot and β in the $(j+1)$ st slot, whereas if $i > j$ then we have

$$\Delta_{i,\alpha}^k \circ \Delta_{j,\beta}^{k-1}(u_1, \dots, u_{k-2})(u_1, \dots, u_{j-1}, \beta, u_j, \dots, u_{i-2}, \alpha, u_{i-1}, \dots, u_{k-2}),$$

with β in the j th slot and α in the i th slot. Thus the double boundary of the standard k -cube consists of two sums, written as formal sums of functions of the variables u_1, \dots, u_{k-2} ,

$$\begin{aligned} \partial^2 \Delta &= \sum_{i=1}^{k-1} \sum_{j=i}^{k-1} (-1)^{i+j+\alpha+\beta} (u_1, \dots, u_{i-1}, \alpha, u_i, \dots, u_{j-1}, \beta, u_j, \dots, u_{k-2}) \\ &\quad + \sum_{i=1}^k \sum_{j=1}^{i-1} (-1)^{i+j+\alpha+\beta} (u_1, \dots, u_{j-1}, \beta, u_j, \dots, u_{i-2}, \alpha, u_{i-1}, \dots, u_{k-2}). \end{aligned}$$

(c) Explain why the second double sum can instead be written as

$$\sum_{j=1}^{k-1} \sum_{i=j+1}^k (-1)^{i+j+\alpha+\beta} (u_1, \dots, u_{j-1}, \beta, u_j, \dots, u_{i-2}, \alpha, u_{i-1}, \dots, u_{k-2}).$$

(d) Convince yourself that it is valid to replace i by $i+1$ in this new second sum, and that doing so gives

$$- \sum_{j=1}^{k-1} \sum_{i=j}^{k-1} (-1)^{i+j+\alpha+\beta} (u_1, \dots, u_{j-1}, \beta, u_j, \dots, u_{i-1}, \alpha, u_i, \dots, u_{k-2}),$$

now with α in the $(i+1)$ st slot.

(e) Convince yourself that it is valid to exchange the roles of i and j , and to exchange the roles of α and β , and that doing so gives

$$- \sum_{i=1}^{k-1} \sum_{j=i}^{k-1} (-1)^{i+j+\alpha+\beta} (u_1, \dots, u_{i-1}, \alpha, u_i, \dots, u_{j-1}, \beta, u_j, \dots, u_{k-2}).$$

This cancels the first sum, so we are done.”

- Page 480: In the first paragraph after the proof ends, change “of k -surfaces” to “of k -surfaces and their boundaries”.