THE LEAST ELEMENT PRINCIPLE AND STRONG INDUCTION

The Least Element Principle says:

> *Let S denote a subset of $\mathbb{N}$. If S is nonempty then S contains a least element.*

The Strong Induction Principle says:

> *Let $P(n)$ be a proposition form over $\mathbb{N}$. If*
>
> *for each $n \in \mathbb{N}$,*
>
> $(P(m)$ *for all* $m \in \mathbb{N}$ *such that* $m < n)$ *implies* $P(n)$
>
> *then $P(n)$ is true for each $n \in \mathbb{N}$.*

This writeup shows that the Least Element Principle and the Strong Induction Principle are equivalent, i.e., they imply one another.

The *strategy* of the argument is more important than its details. Given the hypothesis of the Strong Induction Principle, involving a proposition form $P(n)$, we cunningly

- choose a set S related to $P(n)$, so that the hypothesis on $P(n)$ in Strong Induction translates into the hypothesis on S in the Least Element Principle,
- quote the Least Element Principle to obtain a conclusion about S,
- translate this conclusion back over to $P(n)$, and we have the desired conclusion of Strong Induction.

Similarly, given the hypothesis of the Least Element Principle, involving a set S, we choose a proposition form $P(n)$ related to S so that the hypothesis on S in the Least Element Principle translates into the hypothesis on $P(n)$ in Strong Induction, quote Strong Induction to obtain $P(n)$ for all n, and translate this conclusion back over to S to obtain the desired conclusion of the Least Element Principle.

With the strategy overviewed, we proceed to the argument.

Suppose that the Least Element Principle holds. To establish the Strong Induction Principle in consequence, let $P(n)$ be a proposition form such that for each $n \in \mathbb{N}$, $(P(m)$ for all $m \in \mathbb{N}$ such that $m < n)$ implies $P(n)$; we need to show that $P(n)$ is true for each $n \in \mathbb{N}$. To do so, introduce the set

$$S = \{n \in \mathbb{N} : P(n) \text{ is false}\}.$$

We claim that S has no least element. Suppose the contrary, that S has a least element, n. That is, all $m \in \mathbb{N}$ such that $m < n$ do not lie in S, while n does lie in S. Equivalently, $P(m)$ is true for all $m \in \mathbb{N}$ such that $m < n$, while $P(n)$ is false. This contradicts the assumed character of the proposition form $P(n)$, showing that the supposition that S has a least element is untenable. Now, since S has no least element, the Least Element Principle says that S is empty. This means precisely that $P(n)$ holds for all $n \in \mathbb{N}$, giving the desired conclusion of the Strong Induction Principle.

For the other direction, suppose that the Strong Induction Principle holds. We establish the contrapositive of the Least Element Principle, that any subset S of $\mathbb{N}$ that contains no least element is empty. Let S be such a set and consider the proposition form

$$P(n) = \text{“}n \notin S\text{”}.$$

Now suppose for some generic $n \in \mathbb{N}$ that $P(m)$ is true for all $m \in \mathbb{N}$ such that $m < n$. That is, $m \notin S$ for all $m \in \mathbb{N}$ such that $m < n$. Then also $n \notin S$ because S contains no least element. That is, $P(n)$ is true. This argument shows that the proposition form $P(n)$ satisfies the condition of the Strong Induction Principle, and consequently the Strong Induction Principle says that $P(n)$ is true for each $n \in \mathbb{N}$. This means that $n \notin S$ for each $n \in \mathbb{N}$, i.e., S is empty as desired.