⋆ 1. For each of the polynomial mappings $X \to Y$, describe corresponding ring homomorphisms, $A(Y) \to A(X)$, using the notation of problem 2.

(a)

$$\begin{aligned} \phi : \mathbb{A}^2 &\to \mathbb{A}^3 \\ (x,y) &\mapsto (y - x^2, xy, x^3 + 2y^2) \end{aligned}$$

(b) $X = \mathbb{A}^1$ and $Y = Z((y - x^3, z - xy) \subset \mathbb{A}^3$

$$\begin{aligned} \phi : X &\to Y \\ t &\mapsto (t, t^3, t^4) \end{aligned}$$

⋆ 2. For each of the ring homomorphisms $A(Y) \to A(X)$, describe the corresponding morphism of algebraic sets, $X \to Y$, using the notation of problem 1.

(a)

$$\begin{aligned} \sigma : k[x,y] &\to k[t] \\ x &\mapsto t^2 - 1 \\ y &\mapsto t(t^2 - 1) \end{aligned}$$

(b)

$$\begin{aligned} \sigma : k[s,t,u,w]/(s^2 - w, sw - tu) &\to K[x,y,z]/(xy - z) \\ s &\mapsto xy \\ t &\mapsto yz \\ u &\mapsto xz \\ w &\mapsto z^2 \end{aligned}$$

The morphism constructed here is a mapping of the saddle surface to a surface in $\mathbb{A}^4$.

3. Show that the mapping in 1b, above, is an isomorphism by showing that the induced mappings of coordinate rings is an isomorphism of rings.

⋆ 4. In 2a, above, let C denote the image of the corresponding mapping, $\phi : \mathbb{A}^1 \to \mathbb{A}^2$. Compute $I(C)$. Draw a picture of C letting $k = \mathbb{R}$.

5. We have seen that the parabola, $Z(y - x^2)$, is isomorphic to $\mathbb{A}^1$. Show that the same is not true of the hyperbola, $Z(xy - 1)$. For a challenge consider the circle $Z(x^2 + y^2 - 1)$. Is it ever isomorphic to the parabola? What if the characteristic of k is 2?

6. Zariski closure.

   In a previous problem set, we discussed the Zariski topology on $\mathbb{A}^n$. The closed sets of the topology are taken to be algebraic sets, i.e., sets of the form $Z(I)$ where I is an ideal of $R = k[x_1, \ldots, x_n]$. Consider $\mathbb{A}^n$ with the Zariski topology.

   ⋆ (a) Let $X \subseteq \mathbb{A}^n$. Show that $Z(I(X))$ is the *closure* of set X. This means that $Z(I(X))$ smallest closed set containing X. (Show that if Y is a closed set containing X, then $Y \supseteq Z(I(X))$.)

   (b) What is the closure (in the Zariski topology!) of the open unit disc centered at the origin in $\mathbb{R}^2$?

   (c) What is the closure of the set $\{(n, n) : n \in \mathbb{Z}\}$ in $\mathbb{A}^2_{\mathbb{Q}}$?

   (d) Show that over an algebraically closed field (or just an infinite field), the closure of any nonempty open set of $\mathbb{A}^n$ is $\mathbb{A}^n$, i.e., every nonempty open set is dense. (Again: this is quite a difference from the usual topology in the case of $k = \mathbb{R}$ or $\mathbb{C}$.)

7. Is the composition of two polynomial mappings necessarily a polynomial mapping?

8. Consider the curve $C = Z(y^3 - x^4) \subset \mathbb{A}^2$. Find a mapping $\mathbb{A}^1 \to C$ that is one-to-one and onto but not an isomorphism.

⋆ 9. Suppose that k is algebraically closed, and let $X \subseteq \mathbb{A}^n_k$ be an algebraic set. Show that algebraic sets (respectively, varieties, points) contained in X are in one-to-one correspondence with radical ideals (respectively, prime ideals, maximal ideals) containing $I(X)$.

10. Degeneracy locii.

    (a) Show that the set of singular $n \times n$ matrices forms an algebraic set in $\mathbb{A}^{n^2}$. (An $n \times n$ matrix is singular if it has rank less than n.)

    (b) Fix a nonnegative integer r. Show that $m \times n$ matrices of rank less than r forms an algebraic set in $\mathbb{A}^{mn}$.

    □ (c) Show that the above algebraic sets are varieties.

11. Let $\sigma : B \to A$ be a homomorphism of rings. Show that if $\mathfrak{p} \subseteq A$ is a prime ideal, then $\sigma^{-1}(\mathfrak{p})$ is a prime ideal of B. What if $\mathfrak{p}$ is a maximal ideal?

12. Surjectivity.

    Let $\phi : X \to Y$ be a morphism of algebraic sets, and let $\tilde{\phi} : A(Y) \to A(X)$ be the corresponding homomorphism of coordinate rings.

    (a) Show that if $\tilde{\phi}$ is onto, then X is isomorphic to the algebraic set $Z(\ker \tilde{\phi}) \subseteq Y$.

    (b) Show that if ϕ is onto, then $\ker \tilde{\phi} = (0)$, i.e., $\tilde{\phi}$ is one-to-one.