$\star$ 1. Let $I \subseteq k[x_1, \ldots, x_n]$ be a monomial ideal generated by a set of monomials M. Show that $f \in I$ iff each term of f is divisible by some monomial in M.

$\star$ 2. Calculate the Hilbert function of $I = (x_1x_3, x_1x_4, x_2x_4)$ by hand using the algorithm presented in Lecture 13.

$\star$ 3. Order the terms in the following polynomials using lex, deglex, and revlex ordering, in turn. What is the initial term in each case?

(a) $f = x + 3x - x^2 + z^2 - y^3$.

(b) $g = x^2yz + xy^6 + 2xy^3 - 4x^2y^3z^2$.

$\star$ 4. Give a simple example of an ideal $I = (f_1, \ldots, f_s)$ such that $\text{in}_>(I) \neq (\text{in}_>(f_1), \ldots, \text{in}_>(f_s))$.

$\star$ 5. **Macaulay's theorem.** Let $S = k[x_1, \ldots, x_n]$ with monomial ordering $>$, and let $I \subseteq S$ be an ideal. Let B be the set of monomials of S that are not in $\text{in}_>(I)$. Prove that B is a k-vector space basis for S/I.

Hints:

(a) To show linear independence, let $f = \sum \alpha_i x^{a_i} \in S$ with $\alpha_i \neq 0$ and $x^{a_i} \in B$. Suppose that $f = 0 \in S/I$, i.e., $f \in I$. Now think about the initial term of f.

(b) To show B spans, suppose it does not. Among all elements of S/I not in the span of B, choose one, f, with a smallest initial term. There are two cases to consider depending on whether $\text{in}_>(f) \in \text{in}_>(I)$. In either case, argue there is an element of S/I not in the span of B but with an even smaller initial term.

6. With $I \subset S$ as in the previous problem, Macaulay's theorem says that S/I and $S/\text{in}_>(I)$ are isomophic as k-vector spaces but not necessarily as rings. Now let $S = k[x, y]$ with deglex monomial ordering, and let $I = (y - x^2)$.

(a) Use Macaulay's theorem to exhibit k-bases of S/I and of $S/\text{in}_>(I)$ consisting of the same set of monomials.

(b) Show that S/I and $S/\text{in}_>(I)$ are not isomorphic as rings.

7. Let x^{a_1} be a monomial and $I' = (x^{a_2}, \ldots, x^{a_s})$ be a monomial ideal in $k[x_1, \ldots, x_n]$. In Lecture 13, we considered the mapping which is multiplication by x^{a_1}:

$$S \xrightarrow{\cdot x^{a_1}} S/I'.$$

Show that

$$\ker(\cdot x^{a_1}) = \left(\frac{x^{a_2}}{\gcd(x^{a_1}, x^{a_2})}, \ldots, \frac{x^{a_s}}{\gcd(x^{a_1}, x^{a_s})}\right)$$

where $\gcd(x^a, x^b) = x_1^{\min\{a_1,b_1\}} \cdots x_n^{\min\{a_n,b_n\}}$.