PCMI 2008 Undergraduate Summer School

Lecture 13: Gröbner Bases I

David Perkinson

Reed College
Portland, OR

Summer 2008

Gröbner Bases

Gröbner bases are the central tool of computational algebraic geometry.

Examples of computations for which they are useful:

- the ideal membership problem: $f \overset{?}{\in} I$;
- Hilbert functions;
- resolutions;
- elimination theory;
- finding solutions to systems of equations;
- intersections of ideals.

Main Idea

Reduce all problems in polynomial rings to problems concerning monomials.

Notation

$S = k[x_1, \ldots, x_n]$.

- monomial: $x^a = x_1^{a_1} \cdots x_n^{a_n}$
- exponent vector for x^a: $a = (a_1, \ldots, a_n)$
- degree: $\deg x^a = |a| = \sum_i a_i$
- term: αx^a where $\alpha \in k$
  - Every polynomial is a sum of terms.
- monomial ideal: an ideal generated by monomials
- division of monomials: $x^a | x^b$ if $x^b = f\, x^a$ for some $f \in S$.
  - $x^a | x^b$ iff $b \geq a$, i.e., $b_i \geq a_i$ for all i.

Membership problem

$$1 \overset{?}{\in} (x^2 + y - 3,\ xy^2 + 2x,\ y^3)$$

Yes!

$$\begin{aligned} 1 &= \frac{-1}{27}(y^2 + 3y + 9)(x^2 + y - 3) \\ &\quad - \frac{1}{108}(xy^4 + 3xy^3 + 7xy^2 - 6xy - 18x)(xy^2 + 2x) \\ &\quad + \frac{1}{108}(x^2y^3 + 3x^2y^2 + 9x^2y + 4)y^3 \end{aligned}$$

The problem is easier for monomial ideals...

Proposition

Let $I \subseteq S$ be a monomial ideal generated by a set of monomials M. Then $f \in I$ iff each term of f is divisible by some monomial in M.

Proof.

HW. □

Corollary

Every monomial ideal is generated by a finite set of monomials.

Proof.

Hilbert basis theorem and the above Proposition.

Challenge: Prove this without recourse to the Hilbert basis theorem. (Consider the exponents of any monomial generating set. Which are necessary?) □

Definition

The Hilbert function for a homogeneous ideal $I \subseteq S = k[x_1, \ldots, x_n]$ is the function

$$H_{S/I}(d) = \dim_k S_d / I_d.$$

First Goal

Calculate the Hilbert polynomial of the monomial ideal

$$I = (x^{a_1}, \ldots, x^{a_s}).$$

Write

$$I = (x^{a_1}) + I'$$

where $I' = (x^{a_2}, \ldots, x^{a_s})$, and consider the sequence

$$S(-|a_1|) \xrightarrow{\cdot x^{a_1}} S/I' \xrightarrow{\pi} S/I \longrightarrow 0$$

where $|a_1| = \sum_i a_{1i} = \deg x^{a_1}$.

Claim

The sequence is exact: $\operatorname{image}(\cdot x^{a_1}) = \ker \pi$.

$$I = (x^{a_1}, \ldots, x^{a_s}) = (x^{a_1}) + I'$$

$$S(-|a_1|) \xrightarrow{\cdot x^{a_1}} S/I' \xrightarrow{\pi} S/I \longrightarrow 0$$

Claim: the sequence is exact: image$(\cdot x^{a_1}) = \ker \pi$.

Proof.

- Say $\pi(f) = 0$.
- Pick a representative for f in S. Call it f.
- We may assume f has no terms divisible by $x^{a_2}, \ldots, x^{a_s}$.
- $\pi(f) = 0 \implies f \in (x^{a_1}, \ldots, x^{a_s})$.
- Earlier Proposition implies each term of f is divisible by some x^{a_i}.
- Thus, $x^{a_1} | f$, so $f \in$ image$(\cdot x^{a_1})$.

$\square$

$$S(-|a_1|) \xrightarrow{\cdot x^{a_1}} S/I' \xrightarrow{\pi} S/I \longrightarrow 0$$

HW

$$\ker(\cdot x^{a_1}) = \left(\frac{x^{a_2}}{\gcd(x^{a_1}, x^{a_2})}, \ldots, \frac{x^{a_s}}{\gcd(x^{a_1}, x^{a_s})} \right)$$

where $\gcd(x^a, x^b) = x_1^{\min\{a_1, b_1\}} \cdots x_n^{\min\{a_n, b_n\}}$.

Let $J = \ker(\cdot x^{a_1})$ to get the short exact sequence

$$0 \to S/J(-|a_1|) \xrightarrow{\cdot x^{a_1}} S/I' \longrightarrow S/I \to 0$$

Calculating the Hilbert function of S/I

$$0 \to S/J(-|a_1|) \xrightarrow{\cdot x^{a_1}} S/I' \longrightarrow S/I \to 0$$

Take degrees:

$$0 \to (S/J)_{d-|a_1|} \xrightarrow{\cdot x^{a_1}} (S/I')_d \longrightarrow (S/I)_d \to 0$$

Hilbert function

$$H_{S/I}(d) = H_{S/I'}(d) - H_{S/J}(d - |a_1|).$$

I' and J are monomial ideals with fewer generators. Repeat.

Next Goal

Reduce the problem of calculating the Hilbert function of an arbitrary ideal to the problem of calculating the Hilbert function of a monomial ideal.

Monomial Orderings

Definition

A monomial ordering on $S = k[x_1, \ldots, x_n]$ is a total ordering on the monomials of S such that

1. $x^b > x^a \quad \Longrightarrow \quad x^c x^b > x^c x^a$ for all x^c;
2. 1 is the smallest monomial.

lex: Lexicographical Ordering

$x^b >_{\text{lex}} x^a$ if the left-most nonzero entry of $b - a$ is positive. (Mantra: more of the early variables)

$$x^2 > xy > xz > x > y^2 > yz > y > z^2 > z > 1$$

deglex: Degree Lexicographical Ordering

$x^b >_{\text{deglex}} x^a$ if $|b| > |a|$ or if $|b| = |a|$ and $x^b >_{\text{lex}} x^a$. (Mantra: By degree, breaking ties with lex)

$$x^2 > xy > xz > y^2 > yz > z^2 > x > y > z > 1$$

revlex: Reverse Lexicographical Ordering

$x^b >_{\text{revlex}} x^a$ if $|b| > |a|$ or if $|b| = |a|$ and the right-most nonzero entry of $b - a$ is negative. (Mantra: fewer of the late variables)

$$x^2 > xy > y^2 > xz > yz > z^2 > x > y > z > 1$$

Notes

- From now on, fix a monomial ordering, $>$, on $S = k[x_1, \ldots, x_n]$.
- We will also compare terms: for nonzero $\alpha, \beta \in k$,

$$\alpha x^b > \beta x^a \text{ if } x^b > x^a.$$

Definition

- The initial term of $f \in S$, denoted $\text{in}_>(f)$, is the largest term of f with respect to $>$.
- The initial ideal of an ideal I is the monomial ideal

$$\text{in}_>(I) = (\text{in}_>(f) : f \in I).$$

Macaulay's Theorem

A preliminary

Lemma

Every nonempty set of monomials $\{x^{a_i}\}$ has a least element.

Proof.

Since S is Noetherian the ideal generated by the monomials is generated by a finite subset. Take a least element of this subset. $\square$

Theorem (Macaulay)

Let $I \subseteq S$ be an ideal and $>$ a monomial ordering. Let B be the set of monomials of S not contained in $\mathrm{in}_>(I)$. Then B is a k-vector space basis for S/I.

Proof.

HW (minimal criminal argument). □

Corollary

$$H_{S/I} = H_{S/\mathrm{in}_>(I)}$$

Important Point: We have reduced the problem of computing the Hilbert function of an ideal to that of computing the Hilbert function of a monomial ideal.