Graph Partitions and Free Resolutions of Toppling Ideals

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Abstract

We associate a divisor group with any undirected graph and describe a firing game on the elements in this group. We introduce the graph Laplacian and define the minimal free resolution of the graph toppling ideal. We then study a conjectured form of the foregoing resolution by translating between the algebraic symbols and their combinatorial counterparts in terms of the firing game and graph partitions.

Chapter 1

Graph and Divisors

1.1 Graphs

A **graph** is a tuple $G = (V, E)$ where V is an arbitrary set and E is a set of unordered pairs $\{u, v\}$ such that $u, v \in V$ and $u \neq v$. The elements of V are called **vertices** and the elements of E are called **edges**. Often we write an edge $\{u, v\}$ more concisely as uv. For any vertex v, we call a vertex u a **neighbor** of v if uv is an edge, and define the **degree** of v, denoted by $\deg(v)$, to be the number of neighbors of v, i.e.,

$$\deg(v) = |\{u \in V : uv \in E\}|.$$

A **subgraph** of a graph $G = (V, E)$ is a graph $G' = (V', E')$ where $V' \subseteq V$ and $E' \subseteq E$. In particular, for any $V' \subseteq V$, the **subgraph of G induced by V′** is the subgraph $G' = (V', E')$ where E' is the set of edges in E that contain only vertices in V, i.e.,

$$E' = E \cap (V' \times V') = \{uv : u, v \in V', uv \in E\}.$$

Notice that we may think of "being a subgraph of" as a relation between all graphs, and that this relation induces a partial order on the set of all graphs.

A **path** in a graph is a set of edges of the form $\{v_0v_1, v_1v_2, \cdots, v_{n-1}v_n\}$ where $v_1, v_2, \cdots, v_n$ are distinct vertices of the graph. In this case, we say that the path *connects* v_1 and v_n and that the path has *length* n. A graph is said to be **connected** if every two distinct vertices u, v are connected by a path, and is called **disconnected** otherwise. From this definition it is straightforward to show that a connected subgraph G' of a graph G that is maximal with respect to the subgraph relation is the subgraph of G induced by the vertices of G'. Consequently, any graph maybe divided into disjoint maximal connected subgraphs, that is, these subgraphs share no vertex or edge in common. Each of these subgraphs is called a **connected component** of the graph.

We note that the definitions of connected, disconnected graphs and connected components capture the intuitive idea of a graph being "in one piece" (connected) or "in several pieces" (disconnected and having multiple connected components). Furthermore, as a disconnected graph may be broken down into connected components, for many problems in graph theory it suffices to study connected graphs. Indeed, in this thesis, we shall assume all graphs we deal with to be connected unless otherwise indicated.

We also point out that our definition of a graph can be generalized in many ways. We may allow edges that contain two identical vertices (such edges are called **loops**); we may assign weights to the edges to have a **weighted graph**; further, we may define the edges to be ordered pair of vertices to obtain a **directed graph**. Such generalizations are commonly seen, yet we chose our definition because graphs defined this way possess some particularly nice properties and make many concepts we are interested in easier to define and study.

1.2 Divisors on a Graph

For the rest of the thesis, let $G = (V, E)$ be a finite graph.

A **divisor** on G is a formal sum $D = \sum_{v \in V} a_v v$ where each a_v is an integer. Intuitively, we may think of a divisor as a chip configuration obtained by putting chips on the vertices of the graph, allowing zero or a negative number of chips. The set of all divisors on a graph form a free abelian group under formal addition, denoted by $\mathrm{Div}(G)$. Here "formal sum" and "formal addition" means that we may add the sums in each component, but the components are to remain unrelated, not to be equated or otherwise related. For example, we have $(2u+3v+w)+(2v+2w) = 2u+5v+3w$, but we will not have $u = 2v$ so that $u + 2v = 2u = 4v$. The definition of a free abelian group comes in many versions, some slightly technical, but roughly a free abelian group is just a set of formal sums over some "basis" (in our divisor group, the "basis" is simply V).

We next define a free abelian group identical to the divisor group, called the group of **scripts** and denoted by $\mathrm{Script}(G)$. While we think of divisors as chip configurations on the graph, we shall think of scripts as instructions for an operation on the configurations called **firing**: to fire a single vertex (a script of the form $\sigma = v$), we send one chip from v to each of its neighbors along the edge connecting them; to reverse fire a vertex (a script of the form $\sigma = -v$), reverse the above operation, sending one chip from each of the neighbors of v to v; finally, to fire a generic script

$\sigma = \sum_{v \in V} a_v v$, fire or reverse fire a_v times (fire if $a_v \geq 0$ and reverse fire if $a_v < 0$) the vertex v for each $v \in V$ in any order (it is easy to see that the resulting configuration does not depend on the order of the firing). In formula, we have that the divisor D' obtained by firing the script σ from D is

$$D' = D - \sum_{u \in V} \sum_{uv \in E} \sigma_u (u - v).$$

It is easy to verify that the firing operation induces a group action on $\mathrm{Div}(G)$:

Proposition 1. *Let the* ***Laplacian map*** $\Delta : \mathrm{Script}(G) \to \mathrm{Div}(G)$ *be defined by*

$$\Delta(\sigma) = \sum_{u \in V} \sum_{uv \in E} \sigma_u (u - v),$$

then $\mathrm{Script}(G)$ *acts* $\mathrm{Div}(G)$ *by the group action*

$$\sigma \cdot D = D - \Delta(\sigma).$$

Let D, E be two divisors on a graph G, we say that D **is equivalent to** E, and write $D \sim E$, if there exists a script σ in $\mathrm{Script}(G)$ such that $E = \sigma \cdot D$, i.e., if E may be obtained from D by firing some script. Note that $\sim$ gives an equivalence relation on $\mathrm{Div}(G)$. For any $D \in \mathrm{Div}(G)$, we call the equivalence class D is in the **divisor class** of D and denote it by $[D]$. We define the set of divisor classes to be the **class group** of G and denote it by $\mathrm{Cl}(G)$, i.e., $\mathrm{Cl}(G) = \mathrm{Div}(G)/\mathrm{Image}(\Delta)$. To check well-definedness, note that $\mathrm{Div}(G)$ is abelian and $\mathrm{Image}(\Delta)$ is a subgroup of $\mathrm{Div}(G)$ since Δ is clearly a group homomorphism, therefore $\mathrm{Cl}(G)$ is indeed a group.

The divisor group also has a natural partial order: for $D, E \in \mathrm{Div}(G)$, we say D **is smaller than** E, and write $D \leq E$ if $D_v \leq E_v$ for all $v \in V$,i.e., if D is smaller than E in each component. In particular, we say a divisor D is **effective** if $D \geq \vec{0}$, where $\vec{0} = \sum_{v \in V} 0 \cdot v$. The **linear system** of a divisor D, denoted by $|D|$, is the set of all effective divisors equivalent to D:

$$|D| = \{E \in \mathrm{Div}(G) : E \sim D \text{ and } E \geq \vec{0}\}.$$

Notice that this set might be empty for some divisors on a graph. For example, any divisor $D = \sum_{v \in V} a_v v$ where $\sum_{v \in V} a_v < 0$ has an empty linear system, since firing a script does not change the sum $\sum_{v \in V} a_v$ (the total number of chips on the graph stays the same during firing). The converse is not true, though, since in general we may have a divisor $D = \sum_{v \in V} a_v v$ such that $\sum_{v \in V} a_v > 0$ but $|D| = \emptyset$.

Before we proceed to the next section, we mention that our notions of divisors and firing come from the abelian sandpile model first introduced by Dhar in [3] in relation to the concept of self-organized criticality. This thesis is developed independently of the sandpile model, yet the algebraic and graph theoretic properties of the model may prove useful in understanding our problems in the thesis. For some nice properties of the model relevant to this thesis, we point the reader to [5] and [9].

1.3 Minimal Effective Divisors

Let D be a divisor on G, and let σ be a script in Script(G). We say that D is **effective** with respect to σ if $\sigma \cdot D \geq \vec{0}$, i.e., if firing σ from D results in an effective divisor. In this case we also say that σ is **legal** from D. For a sequence of scripts $S = \sigma_1, \sigma_2, \cdots, \sigma_k$, we say that D is **effective** with respect to S and S is **legal** from D if with $\sigma_0 = \vec{0}$ we have that $D_i = \sum_{j=0}^{i} \sigma_j \cdot D$ is effective with respect to σ_{i+1} for all $i \in \{0, 1, 2, \cdots, k-1\}$, i.e., if in firing $\sigma_1, \sigma_2, \cdots, \sigma_k$ in succession we always obtain effective divisors.

It is known that for any sequence of scripts S, there exists a a unique minimal (in regard to "$\leq$") divisor that is effective with respect to S. Below we shall prove a special case of this claim, where the script corresponds to an ordered partition of the vertices of the graph, as is defined in the next paragraph.

A **k-partition**, or **unordered k-partition**, of the vertex set V is a collection of its disjoint nonempty subsets $\{V_1, V_2, \cdots, V_k\}$ whose union is V. Certainly here $k \leq |V|$. An **ordered k-partition** is an unordered partition with a given order. For any subset $U \subseteq V$, let the **characteristic script** of U be the script $\chi(U) = \sum_{v \in U} v$, then the following holds:

Proposition 2. *Let $S = V_1, V_2, \cdots, V_k$ be an ordered partition of V. There exists a unique* ***minimal effective divisor with respect to*** **S***, that is, a unique divisor $D \in \mathrm{Div}(G)$ such that the script sequence $\chi(V_1), \chi(V_2), \cdots, \chi(V_k)$ is legal from D but not legal from any divisor E where $E \leq D$. This divisor is given by*

$$D = \sum_{i=1}^{k-1} \sum_{v \in V_i} |\{vw \in E : w \in V_{i+1} \cup V_{i+2} \cdots \cup V_k\}| v.$$

i.e., for each $v \in V_i$, D_v is the number of neighbors of v in all V_j such that $i < j \leq k$. In particular, $D = \vec{0}$ when $k = 1$.

Proof. We give a combinatorial proof in terms of chip firing. For D to be effective to $\chi(V_1)$ in the first place, each vertex v in V_1 needs to send a chip to all of its neighbors, but any chip given to a neighbor also in V_1 would be given back by that neighbor, so really the least number of chips need on v equals the number of neighbors of v in all V_j's where $1 < j \leq k$. Next, for $\chi(V_1) \cdot D$ to be effective with respect to $\chi(V_2)$, each vertex v in V_2 needs to send a chip to all of its neighbors, and again we only need to consider its neighbors outside V_2. Further, all neighbors of v that are in some V_j with $j < 2$ (in this case, that is just V_1) will have already sent a chip to v, so the least number of chips need on v in the very beginning equals the number of neighbors of v in all V_j's where $2 < j \leq k$. Easy induction then shows that the given divisor is indeed minimally effective to S, and uniqueness is subsumed in the argument. □

We shall denote the foregoing minimal divisor effective with respect to S by D^S. Following the same idea and adding a little algebraic manipulation, one may prove similarly that such a divisor exists for any sequence of scripts. (In fact, this even works for directed graphs.) However, our main interest will be in script sequences arising from ordered partitions since they have particularly nice combinatorial interpretations. Indeed, one known result asserts that the D^S's where S arises from ordered $|V|$-partitions (i.e., each part is simply a vertex) are exactly the **minimal alive divisors**, minimal divisors that are effective to any script.

The following result is immediate from Proposition 2 and will prove useful later:

Corollary 3. *Let $S = V_1, V_2, \cdots, V_k$ be an ordered k-partition of G and let $S' = V_1, V_2 \cdots, (V_i \cup V_{i+1}), \cdots, V_k$, then*

$$D^S - D^{S'} = \sum_{v \in V_i} |\{vw \in E : w \in V_{i+1}\}| v.$$

Another useful observation is that firing $\chi(V_1)$ from D^S where $S = V_1, V_2, \cdots, V_k$ results in the divisor

$$D' = \sum_{i=2}^{k} \sum_{v \in V_i} |\{vw \in E : w \in V_{i+1} \cup V_{i+2} \cdots \cup V_k \cup V_1\}| v,$$

hence we have the following:

Corollary 4. *Let $S = V_1, V_2, \cdots, V_k$ be an ordered k-partition of G and let $S^{(1)} = V_2, \cdots, V_k, V_1$, the partition obtained from S by one rotation, then $\chi(V_1) \cdot D^S = D^{S^{(1)}}$.*

Chapter 2

Resolution of the Toppling Ideal

Continue to let $G = (V, E)$ be a finite graph. Further, assume that $|V| = n \in \mathbb{Z}$.

2.1 The Graph Laplacian

Let $v_1, v_2, \cdots, v_n$ be an ordering of V. The **degree matrix** of G is the $n \times n$ diagonal matrix Deg where $\mathrm{Deg}_{i,i} = \deg(v_i)$ for all $i \in \{1, 2, \cdots, n\}$. The **adjacency matrix** of G is the $n \times n$ matrix Adj where the $\mathrm{Adj}_{i,j}$ equals 1 if $(v_i, v_j) \in E$ and 0 otherwise. The **Laplacian matrix**, denoted by Δ, is defined to be the difference of the degree matrix and the adjacency matrix:

$$\Delta = \mathrm{Deg} - \mathrm{Adj}.$$

The fact that the notations for the Laplacian matrix and the Laplacian map introduced in Proposition 1 are the same is no coincidence and should cause no confusion, as the Laplacian matrix is indeed the matrix representation of the Laplacian map. One way to see this is to note that the column in the Laplacian corresponding to v_i encodes the instruction for firing v_i: the 1's from the adjacency matrix indicate giving 1 chip to each neighbor of v_i, and the $\deg(v_i)$ from the degree matrix indicates the number of chips v_i should lose in the firing, so that firing v_i from a divisor D results exactly in $D - \Delta(v_i)$. With the above said, we certainly may have defined the Laplacian matrix in terms of the Laplacian map, and indeed we used our current formulation for its simplicity and tangibility. Henceforth we will usually not distinguish between the Laplacian map and the Laplacian matrix.

2.2 Free Resolution of the Toppling Ideal

With the graph Laplacian as a bridge, we make our transition from combinatorics into algebra in this section. We define the graded minimal free resolution of a lattice ideal, a tool commonly used to study the geometry of ideals in algebraic geometry. To get to the definition of the resolution, however, requires developing a series of concepts involving rings and modules. We shall assume some familiarity with ring and module theory from the reader, and we propose that readers who have not seen these definitions before need not worry about mastering the concepts in complete rigor and generality, as our emphasis is simply to understand them in the context of our problem about graph toppling ideals.

We start by letting R be the polynomial ring $\mathbb{C}[x_1, x_2, \cdots, x_n]$. (Recall that $n = |V|$.) Let $v_1, v_2, \cdots, v_n$ be the vertices in V, and define for each $D \in \mathrm{Div}(G)$

$$x^D = \prod_{i=1}^{n} {x_i}^{D_{v_i}}.$$

The **toppling ideal** of G, denoted by I, is (obviously the following is an ideal in R)

$$I = \mathrm{span}_{\mathbb{C}}\{x^D - x^E : D, E \in \mathrm{Div}(G), D \sim E \text{ and } D, E \geq \vec{0}\}.$$

The following sequence of definitions leads to Proposition 10, which establishes S/I as a graded R-module. It is the free resolution of this module that we will be pursuing.

Definition 5 (**Graded Ring**)**.** *A ring S is **graded** by an abelian group A if there are subgroups $S_a \subseteq S$ $(a \in A)$ such that*

$$S = \bigoplus_{a \in A} S_a$$

*as groups and $S_a S_b \subseteq S_{a+b}$ for any $a, b \in A$. An element $f \in S$ is **homogeneous of degree** **a** if $f \in S_a$.*

A simple example of a graded ring is the ring $S = \mathbb{C}[x]$, which is graded by the nonnegative integers, with $S_n = \{cx^n : c \in \mathbb{C}\}$ for all $n \in \mathbb{Z}^+$.

Recall that $\mathrm{Cl}(G) = \mathrm{Div}(G)/\mathrm{Image}(G)$. The following is straightforward:

Proposition 6. *R is graded by $\mathrm{Cl}(G)$.*

Proof. Let $R_D = \mathrm{span}_{\mathbb{C}}\{x^E : E \in |D|\}$ for any $D \in \mathrm{Div}(G)$, then $R = \bigoplus_{D \in \mathrm{Cl}(G)} R_D$. For any $D_1, D_2 \in \mathrm{Div}(G)$ we have $x^{D_1} x^{D_2} = x^{D_1 + D_2}$, hence $R_{D_1} R_{D_2} \subseteq R_{D_1 + D_2}$. □

(As is common practice, we identified a divisor with its equivalence class in the above.)

Definition 7 (**Twist**). *Let S be a ring graded by an abelian group A and let $a \in A$, the a^{th} **twist** of S, denoted by $S(a)$, the same ring S but with a new grading, given by $S(a)_b = S_{a+b}$ for any $b \in A$.*

Continuing with our simple example, the 2^{nd} twist of $\mathbb{C}[x]$ is such that a homogeneous element of degree n in the original ring is of degree $n+2$ in the twisted ring for all $n \in \mathbb{Z}^+$. In general, a twist of a ring simply shifts its grading. The shifting is particularly useful when we wish to guarantee that a mapping between graded modules is homogeneous, as defined below.

Definition 8 (**Graded Module, Homogeneous Mapping**). *Let A be an abelian group and let S be a ring graded by A. A **graded S-module** is a module M with subgroups M_a for $a \in A$ such that*

$$M = \bigoplus_{a \in A} M_a$$

*as groups and $S_a M_b \subseteq M_{a+b}$ for all $a, b \in A$. In particular, a **graded free S-module** is a module M of the form*

$$M = \bigoplus_{a \in A} S(-a)^{\beta_a}$$

as groups, where β_a is a nonnegative integer for each $a \in A$.

*A homomorphism $\phi : M \to N$ of graded free S-modules is **homogeneous of degree zero** if it preserves degree, i.e., if it maps a homogeneous element of degree a in M to a homogeneous element of the degree a as well for any $a \in A$.*

Remark 9. We incorporated twists of the form $S(-a)$ in the definition of graded free modules because, as previously noted, they provide an easy way of rendering a homomorphism homogeneous. Specifically, if a homomorphism $\phi : S \to S$ maps an element of degree $(d-a)$ in S to an element of degree $(d-b)$ in S for any degree d, then ϕ is naturally a homogeneous mapping from $S(-a)$ into $S(-b)$, since an element of degree d in S is of degree $(d-a)$ in $S(-a)$ and of degree $(d-b)$ in $S(-b)$.

Proposition 10. *The quotient group R/I is an R-module graded by* $\mathrm{Cl}(G)$.

Proof. Let $M_{[D]} = \{x^E + I : E \in |D|\}$ for any $D \in \mathrm{Div}(G)$. From the definition of I, it is straightforward to verify that M_D and M_E are disjoint for distinct D, E. It then follows from Proposition 6 that S/I is graded by $\mathrm{Cl}(G)$. □

We have thus established R/I as a free R-module. Defining the graded free resolution of a module now requires one more piece of terminology:

Definition 11 (**Complex and Exact Sequence**). *A* ***complex of*** **S-*****modules*** *is a sequence (finite or infinite) of S-modules F_i and S-module homomorphisms $\phi_i : F_i \to F_{i-1}$ such that $\phi_i \circ \phi_{i+1}$ is zero for all the i's (where ϕ_{i+1}, ϕ_i both exist). The* ***homology*** *of the complex at F_i is the module*

$$\text{Kernel}(\phi_i)/\text{Image}(\phi_{i+1}).$$

The complex is said to be ***exact*** *at F_i if $\text{Kernel}(\phi_i) = \text{Image}(\phi_{i+1})$, i.e., if the homology at F_i is zero. The complex is called an* ***exact sequence*** *if it is exact at all the F_i's (where ϕ_{i+1}, ϕ_i both exist).*

Definition 12 (**Graded Free Resolution**). *A* ***free resolution*** *of an S-module M is an exact sequence*

$$\mathscr{F} : \quad \cdots \longrightarrow F_n \xrightarrow{\phi_n} \cdots \longrightarrow F_1 \xrightarrow{\phi_1} F_0$$

such that $\text{Cokernel}(\phi_1) = M$. The image of the map ϕ_i is called the i-th ***syzygy module*** *of M. The resolution $\mathscr{F}$ is a* ***graded free resolution*** *if S is a graded ring, M and the F_i's are graded, and the ϕ_i's are homogeneous of degree zero. If S is the polynomial ring $\mathbb{C}[x_1, x_2, \cdots, x_n]$, then $\mathscr{F}$ is* ***minimal*** *if $\text{Image}(\phi_i) \subseteq PF_{i-1}$ for all i, where $P = \langle x_1, x_2, \cdots, x_n \rangle$.*

We return to our graded module R/I to end the section. With a little abuse of language, we shall call the free resolution of R/I the **free resolution of the toppling ideal**. To find a resolution of R/I, we may start by finding generators of I and constructing F_1 as a free module where each copy of R corresponds to one of these generators, then map F_1 onto I by sending the generator in each copy of R to its corresponding generator of M, thus obtaining ϕ_1. Next, let M_1 be the kernel of ϕ_1, then M_1 is again an R-module and we may repeat the above procedure with M_1 in place of I, obtaining F_2 and ϕ_2. Continuing in the same fashion gives a free resolution of R/I, and minimality may be guaranteed by choosing minimal sets of generators of I and the kernels M_i. The Hilbert Syzygy Theorem asserts that the resolution is finite if I is finitely generated. Moreover, it is known that R/I has a unique minimal free resolution up to isomorphism. Thus, a unique finite minimal free resolution of R/I exists. A graded free resolution can then be obtained by giving the free modules proper twists as per Remark 9.

2.3 The Resolution from Connected Partitions

In his thesis, Wilmes proves that a minimal set of generators for the toppling ideal I of G can be derived from the minimal effective divisors with respect to the connected 2-partitions of G. Furthermore, in a conjecture a minimal free resolution is proposed in terms of minimal effective divisors with respect to the connected partitions of G. In this section, we state this conjecture and illustrate it with an example.

A k-partition (unordered or ordered) of G is said to be **connected** if the subgraphs of G induced by each part of the partition is connected. We denote the set of all unordered connected k-partitions of G by $\mathcal{P}_k(G)$, and the set of all connected ordered k-partitions of G by $\mathcal{S}_k(G)$. For orderings S_1 and S_2 of $P \in \mathcal{P}_k(G)$, we say S_1 and S_2 are **equivalent**, and write $S_1 \sim S_2$, if S_1 and S_2 have the same underlying unordered partition and D^{S_1} and D^{S_2} are equivalent divisors. Clearly $\sim$ gives an equivalence relation on $\mathcal{S}_k(G)$. As usual, we will denote the equivalence class a partition S is in by $[S]$.

Let $C\mathcal{S}_k(G)$ denote the set of all equivalence classes induced by $\sim$ as above. For each $C \in C\mathcal{S}_k(G)$, pick an arbitrary element $S_C \in C$ as a representative, then any partition in C is obtained by a permutation τ of S_C. We define $\text{sign}(S) = \text{sign}(\tau)$.

Next, we introduce the notion of an ordered refinement. Let $T = V_1, V_2, \cdots, V_k \in \mathcal{S}_k(G)$. If the ordered $(k-1)$-partition $S = (V_1 \cup V_2), V_3, \cdots, V_k$ is also connected, i.e., $S \in \mathcal{S}_{k-1}(G)$, then we say that T is an **ordered refinement** of S, and write $T > S$ or $S < T$.

Recall that we assumed that $G = (V, E)$ where $|V| = n$. For $k \in \{0, 1, \cdots, n-1\}$, let the k-th $\text{Cl}(G)$-graded free R-module of be given by

$$F_k = \bigoplus_{C \in C\mathcal{S}_{k+1}(G)} R(-D^{S_c}).$$

Let e_c be the identity element in $R(-D^{S_c})$. Define the mapping $\bar{\phi}_k : F_k \to F_{k-1}$ by

$$\bar{\phi}_k(e_C) = \sum_{T \in C, \exists S < T} \text{sign}(S)\text{sign}(T) x^{D^T - D^S} e_{[S]} \tag{2.1}$$

Finally, define new homomorphisms ϕ_k from $\bar{\phi}_k$ by normalizing the coefficients, i.e., for $\bar{\phi}_k(e_C) = a_1 m_1 + a_2 m_2 + \cdots + a_l m_l$ with distinct monomials $m_1, m_2, \cdots, m_l$ and nonzero integers $a_1, a_2, \cdots, a_l$, define

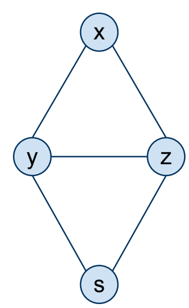
$$\phi_k(e_C) = \frac{a_1}{|a_1|} m_1 + \frac{a_2}{|a_2|} m_2 + \cdots + \frac{a_l}{|a_l|} m_l.$$

Conjecture 13 (**Wilmes**). *The minimal graded free resolution of R/I is given by the following sequence*

$$\mathscr{F}: \quad 0 \longrightarrow F_{n-1} \xrightarrow{\phi_{n-1}} \cdots \longrightarrow F_2 \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0.$$

Notice that the homomorphism $\bar{\phi}_k : R \to R$ indeed maps any element of degree $D - D^T$ in R to an element of degree $D - D^T + D^T - D^S = D - D^S$ in R for $T \in C$. Therefore $\bar{\phi}_k : F_k \to F_{k-1}$, and hence $\phi_k : F_k \to F_{k-1}$, is indeed homogeneous by Remark 9. The minimality of $\mathscr{F}$ is immediate from the fact that for any $S < T$, we have $D^T - D^S \neq 0$ as a consequence of Corollary 3 and the connectedness of T, hence the $[S]$ term of $\phi_k(e_{[T]})$ is not a nonzero scalar. Wilmes also proved that $S/I = \text{Cokernel}(\phi_1)$ and that $\mathscr{F}$ is a complex. Thus, the exactness of $\mathscr{F}$ is what really remains to be proved in the conjecture.

To illustrate how Conjecture 13 works, take $\Gamma = (V, E)$ to be the following graph



We start by computing ϕ_1. Γ has 6 unordered connected 2-partitions, namely

$$P_1 = \{\{x\}, \{y, z, s\}\}, \quad P_2 = \{\{y\}, \{x, z, s\}\}, \quad P_3 = \{\{z\}, \{x, y, s\}\},$$

$$P_4 = \{\{s\}, \{x, y, z\}\}, \quad P_2 = \{\{x, y\}, \{z, s\}\}, \quad P_3 = \{\{x, z\}, \{y, s\}\}.$$

Each P_i gives rise to two ordered partitions of the form $\{A, B\}$ and $\{B, A\}$, which are equivalent by Corollary 4, therefore $C\mathcal{S}_2(\Gamma)$ has 6 elements. Let S_i be P_i with the order as listed (for example, $S_1 = \{x\}, \{y, z, s\}$) and set S_i to be the representative of its class, then each S_i is the ordered refinement of the connected 1-partition $S = V$ (the trivial partition). Let S be the representative of its singleton class, then $\phi_1(e_{S_1})$ is given by

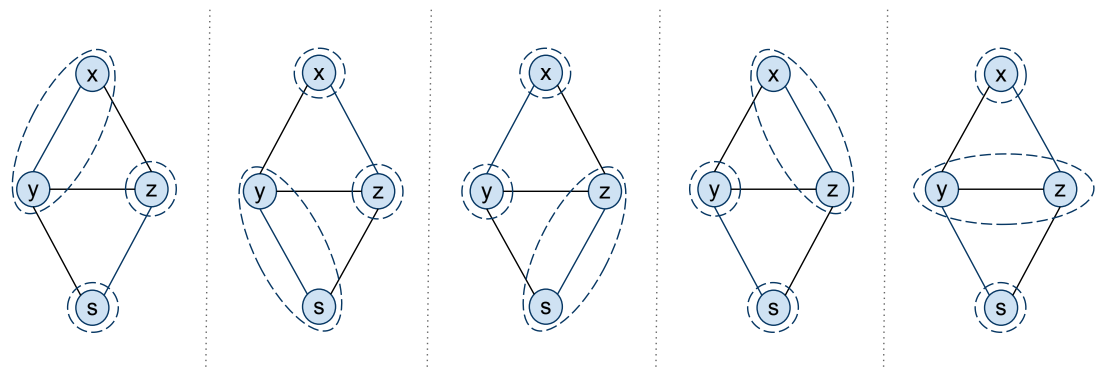
$$\phi_1(e_{S_1}) = \bar{\phi}_1(e_{S_1}) = \sum_{T \in [S_1], \exists U < T} \mathrm{sign}(S)\mathrm{sign}(U) x^{D^T - D^U} e_{[U]} = (1 \cdot 1 \cdot x^2 + (-1) \cdot 1 \cdot yz) e_{[S]}.$$

We may similarly obtain $\phi_1(e_{S_1})$ for all i. They are:

$$\begin{aligned} &\phi_1(e_{S_1}) = (x^2 - yz)e_{[S]}, \quad \phi_1(e_{S_2}) = (y^3 - xzs)e_{[S]}, \quad \phi_1(e_{S_3}) = (z^3 - xys)e_{[S]}, \\ &\phi_1(e_{S_4}) = (s^2 - yz)e_{[S]}, \quad \phi_1(e_{S_5}) = (xy^2 - z^2 s)e_{[S]}, \quad \phi_1(e_{S_6}) = (xz^2 - y^2 s)e_{[S]}. \end{aligned}$$

Consequently, we have $I = \langle x^2 - yz, y^3 - xzs, z^3 - xys, s^2 - yz, xy^2 - z^2 s, xz^2 - y^2 s \rangle$.

To compute ϕ_2, note that Γ has the following five unordered connected 3-partitions:



For each unordered partition $P = \{A, B, C\}$, the six ordered partitions arising from P may at most belong to two distinct equivalence classes, as by Corollary 4 we have

$$A, B, C \sim B, C, A \sim C, B, A$$

$$A, C, B \sim C, B, A \sim B, A, C$$

By straightforward calculation we may see that $[A, B, C] \neq [A, C, B]$ for the leftmost four of the listed partitions, while for the rightmost partition $[A, B, C] = [A, C, B]$.

Let $S'_1 = \{x,y\},\{s\},\{z\}$, arising from the leftmost shown 3-partition, then each $T \in [S'_1]$ has an ordered refinement in one of the $[S_i]$'s. By Corollary 3, we have

$$\phi_2(e_{S'_1}) = 1\cdot(-1)\cdot ye_{[S_3]} + 1\cdot(-1)\cdot se_{[S_5]} + 1\cdot(-1)\cdot z^2 e_{[S_4]} \tag{2.2}$$

Similar calculation then specifies the action of ϕ_2 corresponding to the other seven partition classes S'_i $(i \in \{2,3,\cdots,8\})$ arising from the first four 3-partitions shown.

Now let $S'_9 = \{x\},\{y,z\},\{s\}$, arising from the rightmost shown 3-partition, then $[S'_9]$ consists of all permutations of S'_9. Carefully following Formula 2.1 gives

$$\phi_2(e_{S'_9}) = (s^2 - yz)e_{[S_1]} + (yz - x^2)e_{[S_4]}.$$

Eventually, we obtain that the matrix form of ϕ_2 may be expressed as

$$\phi_2 = \begin{bmatrix} 0 & 0 & -ys & -z^2 & -y^2 & -zs & 0 & 0 & s^2-yz \\ 0 & 0 & 0 & 0 & -z & -x & -s & -z & 0 \\ -y & -s & -x & -y & 0 & 0 & 0 & 0 & 0 \\ -z^2 & -xy & 0 & 0 & 0 & 0 & -xz & -y^2 & yz-x^2 \\ -s & -z & 0 & 0 & x & y & 0 & 0 & 0 \\ 0 & 0 & z & x & 0 & 0 & -y & -s & 0 \end{bmatrix}.$$

We leave it to the reader to verify that the matrix form of ϕ_3 is

$$\phi_3 = \begin{bmatrix} 0 & x & -y & 0 \\ x & 0 & 0 & -y \\ -s & -y & 0 & 0 \\ 0 & 0 & y & s \\ z & s & 0 & 0 \\ 0 & 0 & -s & -z \\ 0 & -z & x & 0 \\ -z & 0 & 0 & x \\ -y & 0 & -z & 0 \end{bmatrix}$$

The number of copies of R in F_3 is 4 because there are 4 equivalence classes of ordered partitions, with $\{x\},\{y\},\{z\},\{s\} \sim \{x\},\{s\},\{y\},\{z\}$ and $\{x\},\{z\},\{y\},\{s\} \sim \{x\},\{s\},\{z\},\{y\}$. The calculation would also require the normalization of $\bar{\phi}$ for the equivalence classes of these two pairs.

Certainly, we note that the calculation of the above two matrices are based on a certain choice of representatives for the partition classes. A different choice may produce entries with different signs.

Working through the foregoing example familiarized us with the mechanics of Conjecture 13, but how are we to interpret the conjecture, especially Formula 2.1? What is the motivation of the conjecture, and how could we prove the conjecture? These questions are not discussed in Wilmes' thesis nor implicit in his partial proof, and we shall investigate them in the next chapter.

Chapter 3

Resolution and Partitions

In this chapter, we examine Conjecture 13 with an emphasis on interpreting Formula 2.1. Recall that the formula states that the identity element e_C in $R(-D^{S_c})$, the copy of R corresponding to a $(k+1)$-partition C of G in the free module F_k , is mapped by ϕ into F_{k-1} by the normalization of the following $\bar{\phi}$:

$$\bar{\phi}(e_C) = \sum_{T \in C, \exists S < T} \operatorname{sign}(S)\operatorname{sign}(T) x^{D^T - D^S} e_{[S]}.$$

Our inquiry consists of three parts accordingly: the first section attempts to motivate the exponent $D^T - D^S$ in the above formula, the second section offers an interpretation of the sign in the formula, and finally we briefly discuss some of our attempts to prove the correctness of the formula and Conjecture 13 in their entirety.

We clarify that by emphasizing on an interpretation of Formula 13, we mean that we shall assume F_1 and ϕ_1 to be known, i.e., we assume we know that the first syzygy module comes from the connected 2-partitions of G. We do so because this knowledge is really our cornerstone that connects the resolution to the combinatorics of the graph, and our aim is to build upon it to explain the higher syzygies of the resolution combinatorially. Also note that from the the syzygies we may recover the free modules of the resolution easily by following the algorithm discussed after Definition 12, therefore once F_1 and F_0 are assumed to be known, then to understand Conjecture 13 it is indeed sufficient to understand Formula 2.1, hence our strategy is well justified.

3.1 The Least Common Multiple View

Following the notations in the previous chapters, let $G = (V, E)$ be a graph. Suppose we know that the toppling ideal I of G is generated by polynomials $g_1, \cdots, g_k$ where the monomial terms in each g_i is of the form $\pm x^{D^S}$ for some ordered 2-partition S. By the remark following Definition 12, for each $f = (f_1, f_2, \cdots, f_k) \in F_1$ we have

$$\phi_1(f) = f_1 g_1 + f_2 g_2 \cdots + f_k g_k. \tag{3.1}$$

To make the sequence $\mathscr{F}$ exact at F_1, we need to find Kernel(ϕ_1), i.e., we need to find f such that the quantity displayed above is 0. But multiplying each g_i by some f_i raises the degree of each monomial in g_i, hence to have $\phi_1(f)$ equal to zero we need to have these resulting higher-degree monomials cancel properly. Since each monomial in the g_i's has degree D^S for some 2-partition S, letting f_i be of the form $\pm x^{D^T - D^S}$ for some 3-partition T's in a partition class then seems a good choice to achieve this goal, for then each monomial in $f_i g_i$ has degree D^T for some T in this partition class and may cancel with each other properly.

To see the above idea in action, recall the action of ϕ_2 on S_1' on the graph Γ. Recall that S_1' is the ordered 3-partition $\{x, y\}, \{s\}, \{z\}$, and Equation 2.2 shows

$$\phi_2(e_{S_1'}) = -y e_{[S_3]} - s e_{[S_5]} - z^2 e_{[S_4]} \tag{3.2}$$

where S_1, S_2, S_3 are representatives of the partition class of ordered 2-partitions refined by S_1'. Specifically, we have

$$S_3 = \{z\}, \{x, y, s\}, \quad S_5 = \{x, y\}, \{z, s\}, \quad S_4 = \{s\}, \{x, y, z\}$$

and they correspond to three generators of I, namely

$$g_3 = z^3 - xys, \quad g_5 = xy^2 - z^2 s, \quad g_4 = s^2 - yz.$$

To view Equation 3.2 in the form of Equation 3.1, we recognize that $e_{S_1'}$ is mapped by ϕ_2 to the element $f = (0, 0, -y, -z^2, -s, 0) \in F_1$, so that we have

$$\begin{aligned} \phi_1(f) &= -y(z^3 - xys) - s(xy^2 - z^2 s) - z^2(s^2 - yz) \\ &= (-yz^3 + yz^3) + (xy^2 s - xy^2 s) + (z^2 s^2 - z^2 s^2) \\ &= 0 \end{aligned}$$

Notice how the f_i's have raised the monomials in g_i's to $\pm xy^2 s, \pm z^2 s^2, \pm yz^3$, each of which correspond to a minimal effective divisor with respect to a 3-partition (obtained by rotations of the parts of S_1') in the partition class of S_1'.

We discuss a simple example to reinforce what we have done so far. Consider the elements $g_1 = xy, g_2 = y^3$ in the polynomial ring $A = \mathbb{C}[x, y]$. Define $\phi : A \times A \to A$ by $\phi(f_1, f_2) = f_1 g_1 + f_2 g_2$, then obviously Kernel$(\phi) = \{(y^2 h, -xh) : h \in \mathbb{C}[x, y]\} = \langle (y^2, -x) \rangle$. Here each element of Kernel(ϕ) is of the form $(y^2 h, -xh)$ because we need $y^2 h$ and $-xh$ to raise xy and y^3, respectively, to get a common multiple of xy and y^3 with opposite signs, just like we need the f_i's to raise the terms in g_i's to their common multiples . Moreover, note that the generator (y^2, x) is such that y^2 and x respectively raises xy and y^3 to just the least common multiple of them. Analogously, in finding generators f for Kernel(ϕ_1) it is reasonable to consider $f \in F_2$ where the f_i's raise the monomials in g_i's to just their least common multiples, which turn out to be (not so surprisingly) of the form $\pm x^{D^T}$ where T is a 3-partition.

If we think of an element f in Kernel(ϕ_1) as giving a relation among the g_i's, the relation being "for what $f_1, f_2, \cdots, f_k$ do we have $\phi_1(f) = f_1 g_1 + f_2 g_2 \cdots + f_k g_k = 0$?", then the foregoing analysis may be further summarized as follows: common refinements of partitions give monomials that are common multiples of the monomials given by those partitions, and relations among monomials are governed by their least common multiples; hence, to find relations among monomials given by 2-partitions, we need to study monomials given by the least common refinement of the 2-partitions, which are 3-partitions. This idea is indeed applicable to finding all syzygies in the resolution of the toppling ideal, explaining roughly why each F_i in the resolution takes its current form.

Indeed, the use of least common multiples is key in calculating resolutions of monomial ideals (the interested reader may refer to Chapter 4 of [6] to see their use in computing cellular resolutions). Unfortunately, our graph toppling ideals are generated by binomials instead of monomials, and the only way of extending the least common multiple methods to binomial generated ideals that we know is through the use of Laurent polynomials, which may cause us to lose our combinatorial information. Partly due to the lack of any exact result, our presentation in this section has been informal, but we hope that our explanation has provided some most basic intuition for understanding Formula 2.1 and Conjecture 13.

3.2 The Partition Complex

We continue to explore the combinatorial content of the maps ϕ_i in the free resolution of the graph toppling ideal. In particular, we will give one explanation of why the signs in front of $x^{D^T - D^S}$ in Equation 2.1 should be of the form they are. Our first step is to construct a complex of R-modules akin to the free resolution of S/I.

Let $\mathcal{T}_k(G)$ be the set of all ordered k-partitions of G, not necessarily connected. For a partition $S = V_1, V_2, \cdots, V_k \in \mathcal{T}_k(G)$ and an integer $0 \leq i \leq k-1$, define

$$S^{(i)} = V_{i+1}, V_{i+2}, \cdots, V_1, V_2, \cdots, V_{i-1}, V_i$$

where addition is modulo k, i.e., $S^{(i)}$ is obtained from i rotations of S , where a **rotation** simply moves the first part in an ordered partition to the last part. For two ordered partitions S, T of G, define $S \simeq T$ if $T = S^{(i)}$ for some i, i.e., if T can be obtained by a rotation of S, then obviously $\simeq$ is an equivalence relation on the partitions of G. Again, for a partition S, we will denote the equivalence class S is in by $[S]$. Thus, if $S \in \mathcal{T}_k(G)$, then $[S]$ is exactly the set $\{S^{(0)}, S^{(1)}, \cdots, S^{(k-1)}\}$.

For any ordered partition S of G, say $S = V_1, V_2, \cdots, V_k$ for some k, let $\widetilde{S}$ be the ordered $(k-1)$-partition obtained by grouping together the first two parts of S, i.e.,

$$\widetilde{S} = (V_1 \cup V_2), V_3, \cdots V_k.$$

Now, for any ordered k-partition S of G that is nontrivial (i.e., $k \geq 2$), define $\varepsilon(S) = x^{D^S}$ and define the map $\varphi_{k-1} : R \to R$ according to the parity of k as follows:

$$\varphi_{k-1}(\varepsilon(S)) = \begin{cases} \displaystyle\sum_{i=0}^{k} x^{D^{S^{(i)}} - D^{\widetilde{S^{(i)}}}} \cdot \varepsilon(\widetilde{S^{(i)}}) & \text{if } k \text{ is odd,} \\ \\ \displaystyle\sum_{i=0}^{k} (-1)^i \cdot x^{D^{S^{(i)}} - D^{\widetilde{S^{(i)}}}} \cdot \varepsilon(\widetilde{S^{(i)}}) & \text{if } k \text{ is even.} \end{cases} \tag{3.3}$$

Let us consider the behavior of $\varphi_{k-2} \circ \varphi_{k-1}$ ($k \geq 3$). Certainly, by the definition of ε, each summand in either of the sum in the above formula is nothing but a copy of $x^{D^{S^{(i)}}}$ for some i. However, notice that if in our computation we do not substitute each ε by the monomial it is and keep them in their formal notation, then $(\varphi_{k-2} \circ \varphi_{k-1})(S)$ is a formal sum where each summand is of the form $f_T \cdot \varepsilon(T)$ where $f_T \in R$ and T is a $(k-2)$-partition obtained by grouping together the first two parts of some $S^{(i)}$ and then grouping together the first two parts of the resulting $(k-1)$-partition again. We claim the following:

Proposition 14. *For every T obtained in the aforementioned way, we have $f_T = 0$.*

Proof. Let $S = V_1, V_2, \cdots, V_k$. And suppose first that k is odd. Every $(k-2)$-partition T obtained in the above way must take one of the following forms:

$$T_1 = (V_{i+1} \cup V_{i+2} \cup V_{i+3}), V_{i+4}, \cdots, V_{i-1}, V_i$$
$$T_2 = (V_{i+1} \cup V_{i+2}), \cdots, (V_{j+1} \cup V_{j+2}), \cdots, V_i$$

To obtain a partition of the form T_1 from S, we must do either of the following

(a) first group together V_{i+1} and V_{i+2} in $S^{(i)}$ to obtain $\widetilde{S^{(i)}}$,
then group together $(V_{i+1} \cup V_{i+2})$ and V_{i+3} in $\left(\widetilde{S^{(i)}}\right)^{(0)}$ to obtain T_1;

(b) first group together V_{i+2} and V_{i+3} in $S^{(i+1)}$ to obtain $\widetilde{S^{(i+1)}}$,
then group together V_{i+1} and $(V_{i+2} \cup V_{i+3})$ in $\left(\widetilde{S^{(i+1)}}\right)^{(k-2)}$ to obtain T_1.

Correspondingly, $\varepsilon(T_1)$ appears in the expansion of $(\varepsilon_{k-1} \circ \varepsilon_k)(S)$ via the following two ways (recall the supposition that k is odd):

(a) since $\varepsilon(T_1)$ appears $\phi_{k-2}\left(\varepsilon(\widetilde{S^{(i)}})\right)$ with the coefficient $x^{D^{\widetilde{S^{(i)}}} - D^{T_1}}$,
where $\varepsilon(\widetilde{S^{(i)}})$ in turn appears in $\phi_{k-1}(S)$ with coefficient $x^{D^{S^{(i)}} - D^{\widetilde{S^{(i)}}}}$;

(b) since $\varepsilon(T_1)$ appears $\phi_{k-2}\left(\varepsilon(\widetilde{S^{(i+1)}})\right)$ with the coefficient $-x^{D^Q - D^{T_1}}$,
where $\varepsilon(\widetilde{S^{(i+1)}})$ in turn appears in $\phi_{k-1}(S)$ with coefficient $x^{D^{S^{(i+1)}} - D^{\widetilde{S^{(i+1)}}}}$;

where $Q = \left(\widetilde{S^{(i+1)}}\right)^{(k-2)} = V_{i+1}, (V_{i+2} \cup V_{i+3}), \cdots, V_i$, hence

$$f_{T_1} = x^{D^{\widetilde{S^{(i)}}} - D^{T_1}} \cdot x^{D^{S^{(i)}} - D^{\widetilde{S^{(i)}}}} - x^{D^Q - D^{T_1}} \cdot x^{D^{S^{(i+1)}} - D^{\widetilde{S^{(i+1)}}}}$$

But the differences in the exponents above are readily computable by Corollary 3:

$$\begin{aligned} D^{\widetilde{S^{(i)}}} - D^{T_1} &= \sum_{v \in V_{i+1} \cup V_{i+2}} |\{vw \in E : w \in V_{i+3}\}| v, \\ D^{S^{(i)}} - D^{\widetilde{S^{(i)}}} &= \sum_{v \in V_{i+1}} |\{vw \in E : w \in V_{i+2}\}| v, \\ D^{Q} - D^{T_1} &= \sum_{v \in V_{i+1}} |\{vw \in E : w \in V_{i+2} \cup V_{i+3}\}| v, \\ D^{S^{(i+1)}} - D^{\widetilde{S^{(i+1)}}} &= \sum_{v \in V_{i+2}} |\{vw \in E : w \in V_{i+3}\}| v. \end{aligned}$$

It then follows easily that f_{T_1} indeed equals 0. A similar proof shows that $f_T = 0$ for a partition T of the form T_2. Finally, a similar proof shows that the proposition also holds when k is even. □

We may now proceed to construct the complex promised at the beginning of the section. Recall that $\mathcal{T}_k(G)$ is the set of all ordered k-partitions of G, and that rotation of parts induces an equivalence relation on $\mathcal{T}_k(G)$. Let $C\mathcal{T}_k(G)$ denote the set of all such equivalence classes. For each $C \in C\mathcal{T}_k(G)$, pick an arbitrary element $S_C \in C$ as a representative. Then, as noted before, C contains exactly the partitions obtained by rotations from S_C. To construct the complex, define

$$M_k = \bigoplus_{C \in C\mathcal{T}_{k+1}(G)} R(-D^{S_c}).$$

for all $k \in \{0, 1, \cdots, n-1\} (n = |V|)$. Let ϵ_c be the identity element in $R(-D^{S_c})$. Define the mapping $\psi_k : M_k \to M_{k-1}$ by

$$\psi_k(\epsilon_C) = \sum_{T \in C} \operatorname{sign}(\widetilde{T}) \operatorname{sign}(T) x^{D^T - D^{\widetilde{T}}} \epsilon_{[\widetilde{T}]}. \tag{3.4}$$

We use Proposition 14 to prove the following:

Theorem 15. *The following sequence is a complex of R-modules:*

$$\mathscr{C} : \quad 0 \longrightarrow M_{n-1} \xrightarrow{\psi_{n-1}} \cdots \longrightarrow M_2 \xrightarrow{\psi_2} M_1 \xrightarrow{\psi_1} M_0.$$

Proof. First observe that if we suppose S is the representative for its equivalence class C, which contain exactly the partitions obtained by rotations from S, then we may write Formula 3.3 more suggestively as

$$\varphi_k(\varepsilon(S)) = \sum_{T \in C} \operatorname{sign}(T) x^{D^T - D^{\widetilde{T}}} \varepsilon(\widetilde{T}).$$

Since we assumed S to be the representative of its class, we have that $\epsilon_C = \varepsilon(S)$. Further notice that indeed $\varepsilon(\widetilde{T}) = \text{sign}(\widetilde{T})\epsilon_{[\widetilde{T}]}$, hence we may write

$$\varphi_k(\epsilon_C) = \sum_{T \in C} \text{sign}(\widetilde{T})\text{sign}(T)x^{D^T - D^{\widetilde{T}}}\epsilon_{[\widetilde{T}]}.$$

That is, ψ_k and ϕ_k are actually identical.

To verify $\psi_{k-1} \circ \psi_{k-2} = 0$ for $k \in \{3, 4, \cdots, n\}$, we may think of each ψ_{j-1} in terms of its matrix, where each column of the matrix specifies how ψ_{j-1} maps a j-partition to the corresponding $(j-1)$-partitions it refines, and each row specifies how the relevant j-partitions are mapped into a $(j-1)$-partition. We may then prove $\psi_{k-1} \circ \psi_{k-2} = 0$ by showing the product of their matrices equals zero, and it is easy to see that verifying that each entry in the product is 0 is exactly equivalent to showing $f_T = 0$ as in Proposition 14. Therefore our sequence $\mathscr{C}$ is a complex. □

Let us compare our new complex $\mathscr{C}$ and the free resolution $\mathscr{F}$ in Conjecture 13. The two main differences between the two are that $\mathscr{F}$ uses only connected partitions of G to produce copies of R in the free modules of $\mathscr{F}$ while $\mathscr{C}$ uses all graph partitions, and that the equivalence relation $\mathscr{C}$ uses to specify its free modules is finer than that in $\mathscr{F}$, that is, $S \simeq T$ implies $S \sim T$ for two ordered partitions S, T. We do not yet known what significance these differences have, but we suspect that considering only connected partitions contributes to the resolution $\mathscr{F}$ being minimal, as in general non-connected partitions will introduce constants into the maps in the complex (by Corollary 3). The choice of the courser equivalence relation is certainly also required for minimality, since the minimal free resolution is unique up to isomorphism and hence the free modules in the resolution should have the correct rank. Experimenting with examples also showed us that once the free modules in $\mathscr{F}$ have been specified by only connected partitions, then the choice of this equivalence relation seems necessary for $\mathscr{F}$ to be a complex in the first place.

We pose a question at the end of this section. For a complete graph K, i.e., a graph where every two distinct vertices are connected by an edge, we have that every partition of K is connected. Do $\mathscr{C}$ and $\mathscr{F}$ coincide for the toppling ideal of K? The ranks of the free modules in the minimal free resolutions of complete graphs seem to form a known sequence ([8]). Can we explain these numbers in terms of graph partitions, and do they tell anything about whether our $\mathscr{C}$ is exact or even gives the minimal resolution of the toppling ideal for the complete graph?

3.3 Algorithms for Computing Resolutions

We briefly discuss the attempts we have made to prove Conjecture 13. The dilemma we have experienced is that in general, it seems very hard to prove exactness of a complex simply by using combinatorics and without depending on any algorithm from algebra, yet known algebraic algorithms that compute resolutions often come in forms that are seemingly unrelated to our combinatorial context. A typical such algorithm we have tried is the well-known Schreyer's algorithm, which uses Gröbner basis to compute resolutions. Unfortunately, the algorithm did not produce free modules compatible with those given by our conjecture on an example we tried. We have also considered pursuing the least common multiple view mentioned earlier, and have wondered whether the complex constructed in the last section is a resolution or may even lead to a minimal resolution after reasonable modification, but it is a shame that we have not made much progress. Near the due date of this thesis, we have been looking at an algorithm in [1]. So far the algorithm has given the same resolution as our conjecture on our example graph Γ, and the use of simplicial complexes in the algorithm is likely to provide a connection to our graph divisors. We hope we will be able to examine this algorithm more carefully somewhere else.

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