Toppling Ideals of M-Matrices

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Livia Xu

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David Perkinson

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Abstract

In this thesis, we look at chip-firing on M-matrices and study the corresponding toppling ideal, generalizing results from [25]. We introduce a class of matrix extensions of M-matrices and show that burning configurations give Gröbner bases of the toppling ideal. Finally, we look at the McKay-Cartan matrix of a faithful complex representation of the cyclic group. We use combinatorial tools to obtain the minimal free resolution of the homogenized toppling ideal when the matrix has no zero entries and we conclude that the ungraded Betti numbers are the Stirling numbers of the second kind.

Introduction

Given an undirected graph G, we can place chips on its vertices and play the *chip-firing* game. The way chips are placed is called a *configuration*. The chips are allowed to disperse along the edges according to certain rules. For example, consider the diamond graph shown in Figure 1. We start with putting four chips on v_2 and one on v_1. The pile of $\bullet$ next to each vertex denotes the chips being placed on that vertex. We then fire one chip along each edge starting at v_2 and end up with a new configuration having two chips on v_1 and one on v_2, v_3, and v_4.

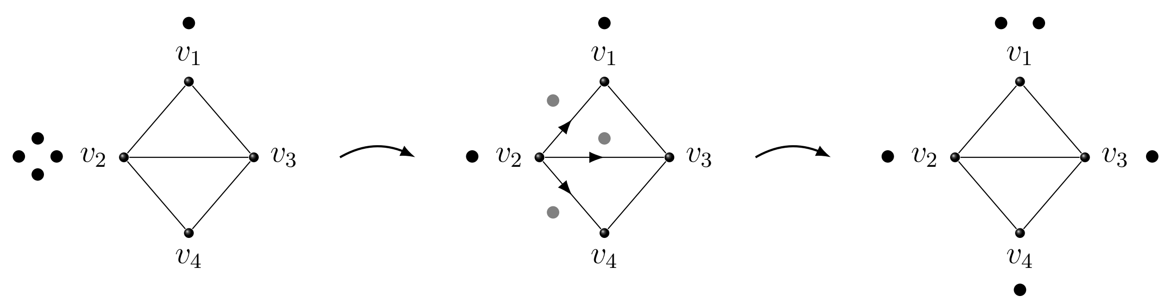

Figure 1: A chip-firing on the diamond graph.

Chip-firing on a graph itself arose as a game worth of studying ([4], [5]). Apart from this, it has arisen naturally under different names in a wide variety of areas of mathematics: as the *group of components* in arithmetic geometry [18], the *Abelian Sandpile Model* in statistical physics [1], in probability theory [17], and in pattern formulation [16]. In combinatorics, it is connected to the theory of parking functions [27]. In algebraic geometry, chip-firing is envisioned as a discrete version of divisor theory on an algebraic curve or Riemann surface [2].

The mysterious firing rules on the graph are encoded in a matrix called the *Laplacian* of a graph as an analogue of the continuous Laplacian operator. In [12] and [14], it is shown that the chip-firing game can be extended to a larger class of matrices called *M-matrices* that has already been used in the study of partial differential equations [26]. In [3] and [13], it is shown that the *McKay-Cartan matrix* of a faithful complex representation of a finite group is an M-matrix. This naturally brings the language of chip-firing to representation theory.

From the point of view of commutative algebra, chip-firing can be interpreted as polynomial division [7]. For example, in the diamond graph, we label the vertices as indeterminates x, y, z, and w instead of v_1, v_2, v_3, v_4 as shown in Figure 2. A

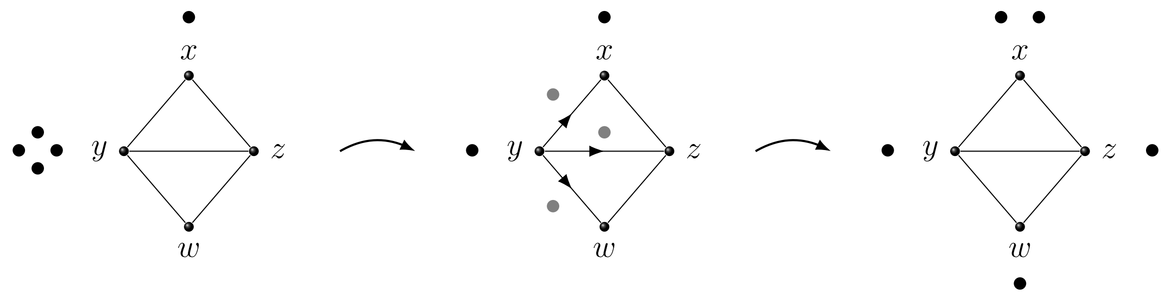

Figure 2: Chip-firing and division by binomial.

configuration of chips gives us a monomial with the exponent of each indeterminate given by the number of chips put on the vertex. The configuration where there are four chips on v_2 and one on v_1 corresponds to the monomial xy^4. Firing chips at v_2 results in a new configuration, which corresponds to the remainder of dividing xy^4 by the binomial $y^3 - xzw$: $xy^4 - xy(y^3 - xzw) = x^2yzw$. The binomial $y^3 - xzw$ is called a *toppling polynomial* and corresponds to the firing rule of v_2 losing three chips and each of v_1, v_3, v_4 gaining one.

The image of the Laplacian of a graph consists of all possible firings of vertices. These firings may be encoded as binomials, as above, and together they form what is known as the *toppling ideal* of the graph in a polynomial ring. These ideals were introduced in [7] and [25]. It is then natural to find the minimal free resolution of the ideal and attach to it algebraic invariants called *Betti numbers*. In [29], John Wilmes began to describe the connection between the combinatorics of the graph and the Betti numbers of the toppling ideal. The project was further developed in [19], [15], and [20].

The goal of this thesis is to develop the theory in [25] and to use it to study the toppling ideals associated with M-matrices. To see the theory in action, we look at a faithful complex representation of the cyclic group that has no zero entries in its McKay-Cartan matrix. We use the combinatorial tools introduced in [24] and [11] and show that the ungraded Betti numbers of the corresponding lattice ideal are the Stirling numbers of the second kind. Some directions for future work include: What are the graded Betti numbers? What happens when there are entries being zero in the McKay-Cartan matrix? How can we apply the results to representations of other finite groups like Abelian groups and symmetric groups?

The first two sections of Chapter 1 generalize the chip-firing language to M-matrices and introduce a class of matrices which extends a given M-matrix through left and right burning scripts. The last section recalls the extended McKay-Cartan matrix and the McKay-Cartan matrix of a faithful complex representation of a finite group as presented in [3]. In Proposition 1.3.3, we show that the extended matrix is a special case of our more general construction.

Chapter 2 studies the lattice ideals corresponding to a given M-matrix and its extension and generalizes relevant results from [25]. We introduce a weighted sandpile monomial ordering and give a Gröbner basis of the lattice ideal corresponding to the extended M-matrix in Corollary 2.2.5. The last section follows [11] and [24] and introduces the Scarf complexes of monomial ideals and lattice ideals. The Scarf complex provides a combinatorial approach to finding minimal free resolutions of generic lattice ideals from a free resolution of its initial ideal, as explained in Theorem 2.3.23 and Corollary 2.3.24.

Chapter 3 looks at faithful complex representations of the cyclic group. The extended McKay-Cartan matrices of these representations are the full Laplacian of some directed multigraph. We work only with the case when the graph is saturated, i.e., when there are edges between each ordered pair of its vertices. Following [27] and [21], Lemma 3.3.6 shows that the Scarf complex of the initial ideal of the lattice ideal corresponding to the extended McKay-Cartan matrix is the barycentric subdivision of the simplex of one dimension lower. Theorem 3.3.7 then gives the minimal free resolution of the lattice ideal, and we conclude that the ungraded Betti numbers are the Stirling numbers of the second kind.

Chapter 1

Chip-firing on M-Matrices

1.1 Chip-Firing on a Graph

Let $G = (\widetilde{V}, E)$ be a directed weighted graph with vertices $\widetilde{V} = \{v_1, \ldots, v_{n+1}\}$ and a weight function $\mathrm{wt}\colon \widetilde{V} \times \widetilde{V} \to \mathbb{N}$. For simplicity, suppose there is no loop in G. The edges form a multiset E with each pair of ordered vertices (v_i, v_j) showing up $\mathrm{wt}(v_i, v_j)$ times. The *(full) Laplacian of* G is the matrix $\widetilde{L} \in M_{(n+1)\times(n+1)}(\mathbb{Z})$ with each entry defined as:

$$\widetilde{L}_{i,j} = \begin{cases} -\,\mathrm{wt}(v_j, v_i) & \text{if } i \neq j; \\ \sum_{j=1}^{n+1} \mathrm{wt}(v_j, v_i) & \text{otherwise.} \end{cases}$$

Let $\mathbb{Z}\widetilde{V}$ denote the free $\mathbb{Z}$-module with generators $\widetilde{V}$. A *divisor* D *on* G is an element of this free module, with $D = \sum_{v\in\widetilde{V}} D(v)v$, $D(v) \in \mathbb{Z}$. There is an isomorphism of free $\mathbb{Z}$-modules, $\mathbb{Z}\widetilde{V} \cong \mathbb{Z}^{n+1}$, sending v_i to the i-th standard basis vector of $\mathbb{Z}^{n+1}$. Thus, we can identify divisors on G with vectors. The *degree of* D is the dot product with the all 1's vector, $\deg(D) \overset{\text{def}}{=} \vec{\mathbf{1}} \cdot D \in \mathbb{Z}$. We say that two divisors D and D' are *linearly equivalent* if $D - D' \in \mathrm{im}_{\mathbb{Z}}(\widetilde{L})$. One can check that linearly equivalent divisors have the same degree.

Fixing a vertex $v_i \in \widetilde{V}$, we obtain the *reduced Laplacian of* G *with respect to* v_i, a matrix $L \in M_{n\times n}(\mathbb{Z})$, by removing the row and column corresponding to the entry $\widetilde{L}_{i,i}$ in $\widetilde{L}$. Since we can always relabel the vertices, we may assume that the reduced Laplacian is always with respect to the last vertex v_{n+1}. Hence, $L \in M_{n\times n}(\mathbb{Z})$ is $\widetilde{L}$ with the last row and last column removed.

Denote the vertex set $\widetilde{V}\backslash\{v_{n+1}\}$ as V. Let $\mathbb{Z}V$ be the free $\mathbb{Z}$-module with generators V. A *chip configuration on* G is a linear combination of vertices $c \in \mathbb{Z}V$, i.e., $c = \sum_{i=1}^{n} c_i v_i$ with $c_i \in \mathbb{Z}$. Similarly, there is an isomorphism of free $\mathbb{Z}$-modules $\mathbb{Z}V \cong \mathbb{Z}^n$ given by sending v_i to the i-th standard basis element of $\mathbb{Z}^n$. Then any configuration can be denoted as a column vector. A configuration c is *nonnegative* if $c_i \geq 0$ for all i.

Given a nonnegative configuration c on G, we can think of it as putting c_i chips on the vertex v_i for each i. A *firing at v_i in the configuration c* results in a new configuration $c' = c - Lv_i$. One can check that firing a sequence of vertices in any order results in the same configuration. The rules of vertices losing and gaining chips through firing are encoded in the columns of the reduced Laplacian. We see that the last vertex v_{n+1} will never fire and that chips disappear along edges leading into v_{n+1}.

In this way, we may consider the diagonal entries of L as the costs of firing. When there are too many chips on a vertex, the chips will topple and redistribute themselves amongst vertices along adjacent edges. If $c_i \geq L_{i,i}$ for some i, then we say c is *unstable* at the vertex v_i, and the firing at v_i in c is *legal*, meaning $c - Lv_i$ is still nonnegative. If there is no unstable vertex in c, then c is a *stable* configuration. We say a vertex is *globally reachable* if there is a directed path from every vertex to it. If v_{n+1} is globally reachable, in which case we call v_{n+1} the *sink vertex*, then through firing unstable vertices in c and subsequent configurations, we will reach at a unique stable configuration on G [25, Section 2]. Such a process is called *stabilization*.

Linear equivalence defines an equivalence relation on the set of divisors, and firing at vertices defines an equivalence relation on the set of chip configurations. For *Eulerian* multigraphs, we have an isomorphism [8, Chapter 12]:

$$\mathbb{Z}^{n+1}/\operatorname{im}_{\mathbb{Z}}(\widetilde{L}) \cong \mathbb{Z} \oplus (\mathbb{Z}^n/\operatorname{im}_{\mathbb{Z}}(L)).$$

This gives us motivation to study divisors on a graph through chip-firing.

Example 1.1.1. Consider the following directed graph G drawn in Figure 1.1, with vertices labeled from v_1 to v_4. We use edges without arrows to indicate that there is an edge in both directions connecting the two vertices. Also, edges with multiplicities greater than 1 are labeled with their multiplicities. The full Laplacian $\widetilde{L}$ and the

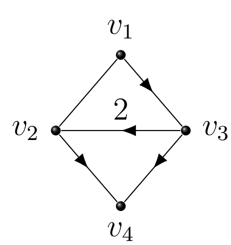

Figure 1.1: The graph G.

reduced Laplacian L of G are:

$$\widetilde{L} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ -1 & 0 & 3 & 0 \\ 0 & -1 & -1 & 0 \end{pmatrix}, \qquad L = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ -1 & 0 & 3 \end{pmatrix}.$$

Start with the configuration $c = (2, 0, 2)$, i.e., put 2 chips on both v_1 and v_3. The stabilization of c goes as shown below, where the label on the arrow indicates the vertex at which a legal firing occurred:

$$(2, 0, 2) \xrightarrow{v_1} (0, 1, 3) \xrightarrow{v_3} (0, 3, 0) \xrightarrow{v_2} (1, 1, 0).$$

1.2 M-Matrices and Extended Matrices

The reduced Laplacian of a graph is a special case of a larger class of matrices, called *M-matrices*. We may generalize the chip-firing language to this class as well.

First, we introduce some notation: We use $\vec{\mathbf{0}}$ and $\vec{\mathbf{1}}$ to denote the vector with its entries being all 0 and the vector with its entries being all 1 of appropriate sizes. Given $c \in \mathbb{Z}^n$, we say $c \geq \vec{\mathbf{0}}$ if $c_i \geq 0$ for all i. Also, we say $c \gneq \vec{\mathbf{0}}$ if $c_i \geq 0$ for all i and if $c \neq \vec{\mathbf{0}}$. Finally, we say $c > \vec{\mathbf{0}}$ if $c_i > 0$ for all i. Note that this is equivalent to say that $c \geq \vec{\mathbf{1}}$.

1.2.1 M-matrices

Definition 1.2.1 (Z-matrix, M-matrix)**.** For a matrix $A \in M_{n\times n}(\mathbb{Z})$, we say A is a *Z-matrix* if $a_{i,j} \leq 0$ for $1 \leq i, j \leq n, i \neq j$. A Z-matrix A is an *M-matrix* if it can be written as $sI_n - B$ for some non-negative matrix B, where I_n is the $n \times n$ identity matrix and s is larger than the absolute values of all eigenvalues of B.

Given a Z-matrix $A \in M_{n\times n}(\mathbb{Z})$, we can associate a directed weighted graph G_A to it. The vertices of G_A are labeled $v_1, \dots, v_n$, and the weights of edges are determined by $\mathrm{wt}(v_i, v_j) = -A_{j,i}$ for any pair of distinct vertices v_i, v_j. It also makes sense to talk about chip-firing on the associated graph G_A: the firing costs are the diagonal entries of A, and the firing rules are given by the columns of A.

Example 1.2.2. Let us see an example of the graph associated to a Z-matrix and see how the language of chip-firing is extended to Z-matrices. Consider

$$A = \begin{pmatrix} 5 & -3 & -2 \\ -2 & 5 & -2 \\ -2 & -3 & 4 \end{pmatrix}.$$

The graph associated to A, G_A, is pictured in Figure 1.2. Note that the costs of firing vertices, i.e., the diagonal entries of A, cannot be inferred from the graph. Start with

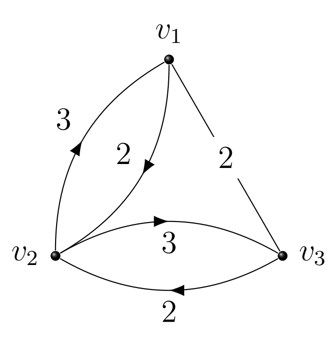

Figure 1.2: G_A, the graph associated to A.

the configuration $c = (1, 3, 5)$, i.e., put 1 chip on v_1, 3 on v_2, and 5 on v_3. A process of stabilizing c via legal firings looks like this:

$$(1,3,5) \xrightarrow{v_3} (3,5,1) \xrightarrow{v_2} (6,0,4) \xrightarrow{v_1} (1,2,6) \xrightarrow{v_3} (3,4,2).$$

Actually, all configurations on A are guaranteed to stabilize since A is an M-matrix, as will be shown in Theorem 1.2.3 (check that A is inverse-positive).

Given a Z-matrix A, we say that A is *avalanche-finite* if every nonnegative configuration on G_A stabilizes. It is well-known in the chip-firing literature (see [14], for example) that the reduced Laplacian of a graph is avalanche-finite. In fact, avalanche-finiteness characterizes non-singular M-matrices.

Theorem 1.2.3 (Characterizations of M-Matrices. [14], [26])**.** *Let $A \in M_{n\times n}(\mathbb{Z})$ be a Z-matrix. Then the following are equivalent:*

1. *A is a non-singular M-matrix.*

2. *The transpose $A^\top$ of A is a non-singular M-matrix.*

3. *A is* inverse-positive*, that is, A^{-1} exists and all entries of A^{-1} are nonnegative.*

4. *A is avalanche-finite.*

5. *There exists $u > \vec{\mathbf{0}}$ with $Au \gneq \vec{\mathbf{0}}$ such that if $(Au)_{i_0} = 0$ for some i_0, then there exist indices $1 \leq i_1, \ldots, i_r \leq n$ with $a_{i_k,i_{k+1}} \neq 0$ for $0 \leq k \leq r-1$ and $(Au)_{i_r} > 0$.*

Hereon, when we speak about configurations on a graph G, we assume that G is the graph associated to some non-singular integer M-matrix $A \in M_{n\times n}(\mathbb{Z})$ and that G has vertices $V = \{v_1, \ldots, v_n\}$.

1.2.2 Burning configurations and the extended matrix

Now we introduce a type of configuration and use it to form an extended matrix for a non-singular M-matrix A, so that the relationship between A and the extended matrix is analogous to that of the reduced Laplacian of a graph and its full Laplacian.

For a configuration c on a graph G, we define its *support* to be the set of vertices v_i such that $c_i > 0$ and denote it as $\mathrm{supp}(c)$.

Definition 1.2.4 (Burning configurations, [25, Definition 2.26])**.** A nonnegative configuration b is a *right burning configuration of* A if it has the following properties:

1. $b \in \mathrm{im}_{\mathbb{Z}}(A)$, $b \neq \vec{\mathbf{0}}$;

2. for all $v \in V$, there exists a directed path to v from some element of $\mathrm{supp}(b)$.

If b is a right burning configuration, we call $\sigma_b = A^{-1}b$ the *script* for b. For $\sigma > 0$, if $A\sigma$ is a right burning configuration, we call σ a *right burning script of* A. If $b'^{\top}$ is a right burning configuration for $A^{\top}$ with script $\sigma'^{\top}$, then b' is a *left* burning configuration of A, and σ' is a *left burning script of* A.

Burning configurations exist by the last characterization of non-singular M-matrices in Theorem 1.2.3, and there is a unique burning configuration with the script being minimal as shown in [22, Theorem 1.38]. To find this configuration, we start with $u = \vec{\mathbf{1}}$ and look at Au. If $(Au)_i < 0$ for some i, then we replace u_i by $u_i + 1$ and repeat the process.

Proposition 1.2.5. *Let $A \in M_{n\times n}(\mathbb{Z})$ be a non-singular M-matrix. Let $\sigma \in \mathbb{N}^n$ be such that $b = A\sigma \gneq \vec{\mathbf{0}}$. Then b is a right burning configuration if and only if $\sigma \geq \vec{\mathbf{1}}$.*

Proof. We will prove both directions by contradiction.

($\Rightarrow$) Suppose that b is a burning configuration, and suppose for contradiction that $\sigma_i = 0$ for some i. Let us look at b_i. Since $A_{i,j} \leq 0$ for $j \neq i$, we have $b_i \leq 0$. But b is a burning configuration, so $b_i = 0$. This implies that $A_{i,j} = 0$ for all $j \neq i$, i.e., there is no edge with tail v_i. Thus, we have $v_i \notin \operatorname{supp}(b)$ and at the same time, v_i is not reachable by $\operatorname{supp}(b)$. We can conclude that b is not a burning configuration, which gives us the contradiction.

($\Leftarrow$) Suppose that $\sigma \geq \vec{\mathbf{1}}$, and suppose for contradiction that there are some vertices not reachable by $\operatorname{supp}(b)$. If there is only one vertex v_i not reachable by $\operatorname{supp}(b)$, then there is no edge with tail v_i, i.e., $A_{i,j} = 0$ for all $j \neq i$. But since $\sigma \geq \vec{\mathbf{1}}$, we have $b_i = A_{i,i} > 0$. This means that $v_i \in \operatorname{supp}(b)$, which contradicts with the assumption.

Now let T denote the set of vertices not reachable by $\operatorname{supp}(b)$, $|T| = k$. From the previous paragraph, we know that $1 < k < n$. Similarly, there is no edge with its head in $V\backslash T$ and its tail in T, i.e., $A_{i,j} = 0$ for all $v_i \in T, v_j \in V\backslash T$. Therefore we can rearrange A into the form

$$A = \begin{pmatrix} B & 0 \\ C & D \end{pmatrix},$$

where B is a $k\times k$-matrix corresponding to the k vertices in T, D is a $(n-k)\times(n-k)$-matrix, and 0 is the $k \times (n-k)$ zero matrix. Since all the vertices in T are not in $\operatorname{supp}(b)$, we have

$$\begin{pmatrix} B & 0 \end{pmatrix} \sigma = \vec{\mathbf{0}}.$$

This means that $\det(B) = 0$, and $\det(A) = \det(B) \cdot \det(D) = 0$. But A is non-singular, so we have a contradiction. $\square$

Definition 1.2.6. Let $A \in M_{n\times n}$ be a non-singular M-matrix, and let $u, v \in \mathbb{N}^n$ be two column vectors such that u is a right burning script of A and w a left burning script of A. Define the (u, w)*-extension of* A, a Z-matrix $\widetilde{A} \in M_{(n+1)\times(n+1)}(\mathbb{Z})$ as follows:

$$\widetilde{A} \stackrel{\text{def}}{=} \left(\begin{array}{c|c} A & -Au \\ \hline -w^{\top}A & w^{\top}Au \end{array}\right).$$

We see that $\widetilde{A}$ has left kernel spanned by $(w,1)$, and right kernel spanned by $(u,1)$. Furthermore, the graph G_A is the subgraph of $G_{\widetilde{A}}$ obtained from removing v_{n+1}.

Remark 1.2.7. This construction was introduced in [6, Section 7], where the authors studied the linear systems on graphs and generalized their results to M-matrices.

Remark 1.2.8. If A is the reduced Laplacian of an undirected graph G, then the $(\vec{\mathbf{1}},\vec{\mathbf{1}})$-extension of A is its full Laplacian. However, if G is directed, then it is not always possible to recover the full Laplacian of G by extending A using the above construction. Although setting $w=\vec{\mathbf{1}}$ gives the last row of the full Laplacian (except for the last entry), it is not guaranteed that the last column of the full Laplacian (except for the last entry) lies in $\mathrm{im}_{\mathbb{Z}}(A)$.

For example, let us return to Figure 1.1, with the full and the reduced Laplacians listed below:

$$\widetilde{L}=\begin{pmatrix} 2 & -1 & 0 & 0\\ -1 & 2 & -2 & 0\\ -1 & 0 & 3 & 0\\ 0 & -1 & -1 & 0\end{pmatrix},\qquad L=\begin{pmatrix} 2 & -1 & 0\\ -1 & 2 & -2\\ -1 & 0 & 3\end{pmatrix}.$$

The full Laplacian has the last column being all zero. But note that since the reduced Laplacian L is nonsingular, it has trivial kernel. Therefore, there is no such $u\geq\vec{\mathbf{1}}$ such that $Lu=\vec{\mathbf{0}}\in\mathbb{N}^n$.

We can also talk about divisors on $G_{\widetilde{A}}$: A *divisor* D *on* $G_{\widetilde{A}}$ is an element of the free module $\mathbb{Z}\widetilde{V}$, where $\widetilde{V}=V\cup\{v_{n+1}\}$, and we can consider divisors as column vectors. The *degree* of a divisor D is the dot product $(w,1)\cdot D\in\mathbb{Z}$. We say two divisors on $G_{\widetilde{A}}$, D and D' are *linearly equivalent* if $D-D'\in\mathrm{im}_{\mathbb{Z}}(\widetilde{A})$. One can check that the degree is an invariant under this equivalence relation, exactly because the left kernel of $\widetilde{A}$ is spanned by $(w,1)$.

Example 1.2.9. Let us recall Example 1.2.2, where we looked at the graph associated to an M-matrix. Here we look at its extended matrix using its minimal left and right burning scripts. Let

$$A=\begin{pmatrix} 5 & -3 & -2\\ -2 & 5 & -2\\ -2 & -3 & 4\end{pmatrix}.$$

Its minimal left burning script is $w=(6,8,7)$, and the minimal right burning script is $u=(7,6,8)$. Thus, the (u,w)-extension of A is shown below, and the graphs associated to A and $\widetilde{A}$ are pictured in Figure 1.3.

$$\widetilde{A}=\begin{pmatrix} 5 & -3 & -2 & -1\\ -2 & 5 & -2 & 0\\ -2 & -3 & 4 & 0\\ 0 & -1 & 0 & 6\end{pmatrix}.$$

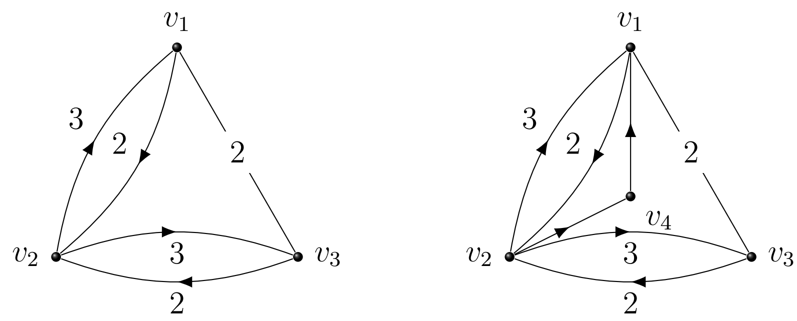

Figure 1.3: Graphs associated to A and $\widetilde{A}$.

We end this section by proving another nice property of the extended matrix.

Proposition 1.2.10. *Let $A \in M_{n\times n}(\mathbb{Z})$ be a non-singular M-matrix, and let u, w be such that we can form $\widetilde{A}$, the (u, w)-extension of A. Then in the graph $G_{\widetilde{A}}$, the vertex v_{n+1} is a sink vertex.*

Proof. Recall that v_{n+1} being a sink vertex means that for each $1 \leq i \leq n$, there is a directed path from v_i to v_{n+1}. Suppose for contradiction that there is some v_i that has no directed path to v_{n+1}. If this is the only vertex with this property, then $A_{j,i} = 0$ for all $j \neq i$, and the i-th entry of $w^T A$ will always be positive. This means that in $G_{\widetilde{A}}$, we have an edge going from v_i to v_{n+1}, which contradicts the assumption. Therefore, there is some v_j such that: 1) $i \neq j$, 2) $A_{j,i} \neq 0$, and 3) there is no directed path going from v_j to v_{n+1}.

We repeat this process on all such v_j and continue until we cannot reach any new vertices. Let T denote the set of vertices reached by this process, and let $k = |T|$. Note that $k < n$ since $w^\top A \neq \vec{\mathbf{0}}$. Also note that there is no edge going from vertices in T to vertices in $V \backslash T$. Thus we may rearrange A into the form

$$A = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix},$$

where B is a $k \times k$ matrix corresponding to the k vertices in T, D is a $(n-k)\times(n-k)$ matrix, and 0 is the $(n-k) \times k$ zero matrix.

All vertices in T do not have a directed path to v_{n+1}, and in particular, there is no edge going from vertices in T to v_{n+1}. Therefore, we have that

$$w^\top \begin{pmatrix} B \\ 0 \end{pmatrix} = \vec{\mathbf{0}}.$$

That is to say, some nontrivial linear combination of the rows of B gives us $\vec{\mathbf{0}}$. Thus, $\det(B) = 0$. Using the rearrangement of A, we have $\det(A) = \det(B) \cdot \det(D) = 0$. This contradicts the assumption that A is non-singular. Thus, there must be a directed path from v_i to v_{n+1}, making v_{n+1} a sink vertex. $\square$

1.3 McKay-Cartan Matrix of a Representation

This section primarily follows [3, Section 5] and [13].

Definition 1.3.1 (McKay-Cartan Matrix)**.** Let G be a finite group. Let $\gamma\colon G \to \mathrm{GL}_m(\mathbb{C})$ be a complex representation of G of dimension m and denote its character by $\chi_\gamma\colon G \to \mathbb{C}$. Let $\rho_1, \ldots, \rho_{n+1}$ denote all the inequivalent irreducible complex representations of G, with ρ_{n+1} being the trivial representation. Use $\chi_1 \ldots, \chi_{n+1}\colon G \to \mathbb{C}$ to denote their characters. For $1 \leq i \leq n+1$, we can decompose each product $\chi_\gamma \cdot \chi_i$ as a sum of irreducible representations, i.e.,

$$\chi_\gamma \cdot \chi_i = \sum_{j=1}^{n+1} m_{i,j}\chi_j.$$

Record the coefficients as the matrix $M = (m_{i,j}) \in M_{(n+1)\times(n+1)}(\mathbb{Z})$. Define the *extended McKay-Cartan matrix* as $\widetilde{C} \overset{\text{def}}{=} mI_{n+1} - M^\top$, where I_{n+1} is the identity matrix of size $n+1$, and define the *McKay-Cartan matrix* as the submatrix $C \in M_{n\times n}(\mathbb{Z})$ of $\widetilde{C}$ by removing the row and column corresponding to χ_{n+1}, i.e., C is the submatrix obtained by removing the last row and the last column.

Theorem 1.3.2 ([3, Theorem 1.2])**.** *If a complex representation of a finite group is faithful, then its McKay-Cartan matrix is avalanche-finite.*

The *McKay quiver of* γ is the directed graph that has n vertices that correspond to $\chi_1, \ldots, \chi_n$ and has $m_{i,j}$ edges from χ_i to χ_j for any pair of i, j. We can perform chip-firing on these quivers, with firing rules encoded in the columns of C.

Proposition 1.3.3 ([3, Proposition 5.6, Proposition 5.14])**.** *Following the notation as above, let*

$$d = (\dim(\rho_1), \ldots, \dim(\rho_n)).$$

Then:

1. *The vector* $(d, 1)$ *spans both the right and left kernels of* $\widetilde{C}$*;*
2. d *is both a right and a left burning script for* C*;*
3. *The* (d, d)*-extension of* C *is* $\widetilde{C}$*.*

Proof. The first part is [13, Proposition 8]. In [3, Proposition 5.14], it has been shown that d is a right burning script for C. The last part follows through a computation using these two results and the definition of the extended matrix. Finally, d is a left burning configuration by Proposition 1.2.5. □

Thus, we focus on a faithful representation $\gamma\colon G \hookrightarrow \mathrm{GL}_m(\mathbb{C})$, in which case its McKay-Cartan matrix C is a non-singular M-matrix, and we consider its (d, d)-extension, where $d = (\dim(\rho_1), \ldots, \dim(\rho_n))$.

Remark 1.3.4. It is often seen in the literature that the trivial representation is denoted as ρ_0 and appears first amongst the irreducible representations. We decide to put it in the end so that the lattice ideals associated to $\mathrm{im}_{\mathbb{Z}}(C)$ and $\mathrm{im}_{\mathbb{Z}}(\widetilde{C})$ behave well under the monomial ordering we choose. It is also common in the literature that the extended McKay-Cartan matrix is defined as $mI_{n+1} - M$, where here we are taking its transpose in order to use the columns as firing rules on the McKay quiver.

Example 1.3.5. Let C_3 denote the cyclic group of order 3 with generator g. It has three inequivalent irreducible complex representations ρ_1, ρ_2, ρ_3 given by sending the generator of C_3 to $e^{2\pi i/3}$, $e^{4\pi i/3}$, and 1 respectively. Let $\gamma\colon C_3 \to \mathrm{GL}_3(\mathbb{C})$ be the representation that sends the generator to the permutation matrix of (1 2 3) as shown below:

$$g \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

and let χ_γ denote its character. The character table of C_3 along with γ is shown in Table 1.1.

	e	g	g^2
ρ_1	1	$e^{2\pi i/3}$	$e^{4\pi i/3}$
ρ_2	1	$e^{4\pi i/3}$	$e^{2\pi i/3}$
ρ_3	1	1	1
γ	3	0	0

Table 1.1: Character table of C_3.

Now we form the extended McKay-Cartan matrix $\widetilde{C}$ for γ. First we decompose γ into $\gamma \cong \rho_1 \oplus \rho_2 \oplus \rho_3$. Now note that $\chi_\gamma \cdot \chi_1 = \chi_\gamma \cdot \chi_2 = \chi_\gamma \cdot \chi_3 = \chi_1 + \chi_2 + \chi_3$. So:

$$M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad \widetilde{C} = 3I_3 - M = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

The McKay quiver of γ is shown in Figure 1.4. The dimensions of all irreducible representations of C_3 are 1, which gives us $d = (1, 1)$. The corresponding configurations Cd and $d^\top C$ are the same, both being $(1, 1)$.

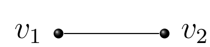

Figure 1.4: The McKay quiver of γ.

Example 1.3.6. Let S_4 denote the symmetric group on 4 letters. The four nontrivial irreducible representations have characters as shown in Table 1.2, where ρ_1 is the sign representation, ρ_3 is the standard representation, ρ_4 is the tensor product $\rho_1 \otimes \rho_3$, and ρ_2 is the pullback of the standard representation of S_3.

	e	$(1\ 2)$	$(1\ 2\ 3)$	$(1\ 2\ 3\ 4)$	$(1\ 2)(3\ 4)$
χ_1	1	-1	1	-1	1
χ_2	2	0	-1	0	2
χ_3	3	1	0	-1	-1
χ_4	3	-1	0	1	-1
χ_5	1	1	1	1	1

Table 1.2: Character table of S_4.

Let γ be the standard representation ρ_3. The matrices M, $\widetilde{C}$, and C are:

$$M = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad \widetilde{C} = \begin{pmatrix} 3 & 0 & 0 & -1 & 0 \\ 0 & 3 & -1 & -1 & 0 \\ 0 & -1 & 2 & -1 & -1 \\ -1 & -1 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 3 \end{pmatrix}, \quad C = \begin{pmatrix} 3 & 0 & 0 & -1 \\ 0 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ -1 & -1 & -1 & 2 \end{pmatrix}.$$

The McKay quiver for γ is shown in Figure 1.5. We have $d = (1, 2, 3, 3)$, and the corresponding burning configurations are both $(0, 0, 1, 0)$.

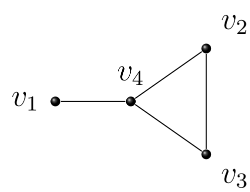

Figure 1.5: The McKay quiver of γ.

Chapter 2

M-Matrices and Lattice Ideals

2.1 Lattice Ideal from a Non-singular M-Matrix

First we introduce some notation. Let $R = \mathbb{C}[x_1, \ldots, x_n]$ and $S = \mathbb{C}[x_1, \ldots, x_{n+1}]$, where $x_1, \ldots, x_{n+1}$ are indeterminates. For $u \in \mathbb{N}^n$, we use x^u to denote the monomial

$$x^u = \prod_{i=1}^{n} x_i^{u_i} \in R.$$

For $v \in \mathbb{N}^{n+1}$, we use $\widetilde{x}^v$ to denote the monomial

$$\widetilde{x}^v = \prod_{i=1}^{n+1} x_i^{v_i} \in S.$$

Let $u \in \mathbb{Z}^n$. Define $u^+, u^- \in \mathbb{N}^n$ as follows:

$$u_i^+ = \begin{cases} u_i & \text{if } u_i \geq 0 \\ 0 & \text{otherwise} \end{cases}, \qquad \text{and} \qquad u_i^- = \begin{cases} -u_i & \text{if } u_i \leq 0 \\ 0 & \text{otherwise} \end{cases}.$$

One can see that $u = u^+ - u^-$.

Definition 2.1.1 (Lattice Ideal)**.** A *lattice* $\mathcal{L} \subseteq \mathbb{Z}^n$ is a submodule of $\mathbb{Z}^n$. It can be written as the image of some integer matrix L with n rows, i.e., $\mathcal{L} \cong \mathrm{im}_{\mathbb{Z}}(L)$. In this case, $\mathcal{L}$ is generated by the columns of L. The *lattice ideal associated to* $\mathcal{L}$, $I_{\mathcal{L}}$, or $I(L)$, is the binomial ideal defined as

$$I(L) = I_{\mathcal{L}} \stackrel{\text{def}}{=} \langle x^u - x^v \mid u, v \in \mathbb{N}^n,\, u = v \bmod \mathcal{L} \rangle \subseteq R.$$

Let I_L denote the ideal generated by binomials $x^{c^+} - x^{c^-}$, where c ranges through the columns of L.

Lemma 2.1.2 ([11, Lemma 7.6])**.** *Using the notation above, the lattice ideal $I_{\mathcal{L}}$ is computed from I_L by taking the saturation with respect to the product of all the variables:*

$$I_{\mathcal{L}} = (I_L : \langle x_1 \cdots x_n \rangle^{\infty}),$$

which by definition is the ideal $\{f \in R \mid (x_1 \cdots x_n)^m f \in I_L \text{ for some } m > 0\}$.

Saturation of an ideal is hard to compute. However, when L is a non-singular M-matrix, we can compute $I_{\mathcal{L}}$ from I_L by adding an additional generator.

Definition 2.1.3 (Toppling polynomial)**.** Let $A \in M_{n\times n}(\mathbb{Z})$ be an M-matrix with columns $c_1, \dots, c_n$. For $1 \le i \le n$, define the *i-th toppling polynomial* of A to be

$$t_i \stackrel{\text{def}}{=} x^{c_i^+} - x^{c_i^-} = x_i^{A_{i,i}} - \prod_{j\neq i} x_j^{-A_{j,i}}.$$

Proposition 2.1.4 (c.f. [25, Proposition 4.2])**.** *Let $A \in M_{n\times n}(\mathbb{Z})$ be a non-singular M-matrix, and $b \in \mathbb{N}^n$ any burning configuration on A. Then the lattice ideal associated to its image, $I(A)$, is generated by the toppling polynomials $\{t_i\}_{i=1}^n$ and $x^b - 1$, i.e., $I(A) = I_A + \langle x^b - 1\rangle$.*

Proof. Let $J = I_A + \langle x^b - 1\rangle$. Since $b \in \operatorname{im}_{\mathbb{Z}}(A)$ and $b \ge \vec{\mathbf{0}}$, it is clear that $x^b - 1 \in I(A)$ and thus $J \subseteq I(A)$. By Lemma 2.1.2, it suffices to show that J is already saturated with respect to the ideal generated by the product $x_1 \cdots x_n$. Suppose there is $f \in R$ such that $(x_1 \cdots x_n)^k f \in J$ for some $k > 0$. We will show that $f \in J$ by showing $f = 0 \mod J$.

For each $m > 0$, consider the monomial x^{mb}. Note that mb is a configuration on G_A. If it is unstable at vertex v_i, then firing at v_i results in the configuration $mb - Av_i$, which corresponds to a monomial equivalent to x^{mb} modulo J. During the process of stabilizing mb, we obtain monomials that are equivalent to x^{mb} modulo J.

Recall that by definition, every vertex of G_A is connected by a directed path from some vertex in $\operatorname{supp}(b)$. Thus by taking m large enough and firing appropriate vertices, we arrive at a monomial x^γ which is equivalent to x^{mb} and $\gamma = \delta + (k, k, \dots, k)$ for some $\delta \ge 0$. Using the fact that $x^{mb} = 1 \bmod J$, we have, modulo J,

$$\begin{aligned} 0 &= (x_1 \cdots x_n)^k f \\ &= x^\delta (x_1 \cdots x_n)^k f \\ &= x^\gamma f \\ &= x^{mb} f \\ &= f. \end{aligned}$$

□

In order to better understand the lattice ideal $I(A)$, we want to find a Gröbner basis for it. It turns out that by imposing a weighted sandpile monomial ordering, right burning configurations of A give us Gröbner bases. This will be shown in Proposition 2.1.16.

Definition 2.1.5. A *monomial ordering*, or *term order*, $<$ on R is a total order of monomials that is *multiplicative*, i.e., $x^a < x^b$ if and only if $x^{a+c} < x^{b+c}$ for all $a, b, c \in \mathbb{N}^n$, and is *Artinian*, i.e., $1 < x^a$ for all $a \neq \vec{\mathbf{0}}$.

The following are some common monomial orderings on a polynomial ring:

- The *lexicographic ordering*, `lex`, is defined as $x^a > x^b$ if the left-most nonzero entry of $a - b$ is positive (i.e., more of the earlier indeterminates is larger);

- The *degree lexicographic ordering*, `deglex`, is defined as $x^a > x^b$ if $\deg(x^a) = \vec{\mathbf{1}} \cdot a > \deg(x^b)$ or if $\deg(x^a) = \deg(x^b)$ and the left-most nonzero entry of $a - b$ is positive (i.e., first order by degree and then break ties with `lex`);

- The *reverse lexicographic ordering*, `revlex`, is defined as $x^a > x^b$ if the right-most nonzero entry of $a - b$ is negative (i.e., less of the later indeterminates is larger).

- The *degree reverse lexicographic ordering*, `grevlex`, is defined as $x^a > x^b$ if $\deg(x^a) > \deg(x^b)$ or if $\deg(x^a) = \deg(x^b)$ and the right-most nonzero entry of $a - b$ is negative (i.e., first order by degree and then break ties with `revlex`);

- Given a weight vector $w \in \mathbb{N}^n$, we can define a *w-weighted order*. For any given monomial x^a, instead of looking at its degree, we look at its *w-weighted degree*,

$$\deg(x^a) \stackrel{\text{def}}{=} \deg_w(x^a) \stackrel{\text{def}}{=} w \cdot a.$$

  To compare two monomials x^a and x^b, we first compare their w-weighted degrees, and break ties using some other monomial ordering.

Definition 2.1.6. For a polynomial $f \in R$, its *leading term*, $\mathrm{LT}(f)$, is the term with the largest monomial. For an ideal $I \subseteq R$, its *initial ideal*, $\mathrm{In}(I) = \langle \mathrm{LT}(f) \mid f \in I \rangle$, is the ideal generated by the leading terms of its elements.

Remark 2.1.7. Fix a term order on R. Let $I \subseteq R$ be an ideal with a set of generators $\{f_1, \dots, f_n\}$. Note that the ideal $\langle \mathrm{LT}(f_i) \mid i = 1, \dots, n \rangle$ is not always the same as the initial ideal $\mathrm{In}(I)$. For example, consider $\mathbb{C}[x, y]$ with `grevlex`. Let $f_1 = x^2 + 2xy^2$, $f_2 = xy + 2y^3 + 1$. Note that $x \in I = \langle f_1, f_2 \rangle$ since $x = xf_2 - yf_1$. Thus $x \in \mathrm{In}(I)$. However, $\langle \mathrm{LT}(f_1), \mathrm{LT}(f_2) \rangle = \langle 2xy^2, 2y^3 \rangle$. It is obvious that x is not in this ideal.

Definition 2.1.8 (Gröbner basis)**.** Fix a monomial ordering on R, and let I be an ideal. A finite generating set $\Gamma \subseteq I$ of I is a *Gröbner basis* with respect to the given ordering if $\langle \mathrm{LT}(g) \mid g \in \Gamma \rangle = \mathrm{In}(I)$.

Fix a monomial ordering on R. Let $I \subseteq R$ be an ideal, and let $\Gamma = \{g_1, \dots, g_n\}$ be a Gröbner basis for I. Then for any $f \in I$, we can write f uniquely as

$$f = \sum_{i=1}^{n} s_i g_i + f'$$

for some $s_i \in R$, such that f' has no term divisible by any of the leading terms of g_i's. We call the remainder f' the *reduction of f by* Γ. Note that the expression may not be unique if we do not use a Gröbner basis (see [9, Chapter 2], for example).

Definition 2.1.9 (S-polynomial)**.** Fix a monomial ordering on R, and let $f, g \in R$. Define the S-polynomial for the pair (f, g) to be

$$S(f,g) = \frac{\operatorname{lcm}(\operatorname{LT}(f), \operatorname{LT}(g))}{\operatorname{LT}(f)} f - \frac{\operatorname{lcm}(\operatorname{LT}(f), \operatorname{LT}(g))}{\operatorname{LT}(g)} g.$$

Proposition 2.1.10. *Fix a monomial ordering on R, and let I be an ideal. The following are equivalent for a finite subset Γ of I:*

1. *Γ is a Gröbner basis for I with respect to the given ordering.*
2. *Each $f \in I$ can be reduced to 0 by Γ.*
3. *Γ generates I, and for each pair $g, g' \in \Gamma$, the S-polynomial, $S(g, g')$ can be reduced to 0 by Γ.*

For proofs and a more detailed exposition, see [10, Chapter 15].

Remark 2.1.11. The last criterion in Proposition 2.1.10 is the *Buchberger's algorithm* for computing a Gröbner basis for I from a generating set X. Namely, we compute $S(f, g)$ for each pair $f, g \in X$, add it to X if it cannot be reduced to 0 by X, and compute the S-polynomials again. The process eventually stops.

Let $A \in M_{n\times n}(\mathbb{Z})$ be a non-singular M-matrix, and $\widetilde{A}$ its (u, w)-extension. Now we introduce a *weighted sandpile monomial ordering* on R, and show that the right burning scripts of A give Gröbner bases with respect to this ordering.

The monomial ordering is a w-weighted `grevlex` order, where we first compare the w-weighted degree and break ties by `revlex` using the following order on indeterminates: we say $x_i > x_j$ if the length of the shortest path from v_i to the sink v_{n+1} is longer than that of v_j. Recall Proposition 1.2.10, where we showed that in $G_{\widetilde{A}}$, the graph associated to $\widetilde{A}$, there is a directed path to v_{n+1} from all vertices v_i. If the two paths are of the same length, then we make an arbitrary choice for which of x_i or x_j is larger.

In this way, after comparing the total degrees of monomials, we have ordered the indeterminates $x_1, \ldots, x_n$. When comparing two monomials x^a and x^b, we say $x^a > x^b$ if $w \cdot a > w \cdot b$, or if $w \cdot a = w \cdot b$ and there is i such that $(a-b)_i < 0$ and x_i is the smallest amongst all x_j's with $(a-b)_j \neq 0$.

Remark 2.1.12. If A is the reduced Laplacian of some graph and we choose $w = \vec{\mathbf{1}}$, then the w-weighted sandpile monomial ordering is a *sandpile monomial ordering* introduced in [25, Section 5].

Proposition 2.1.13 (c.f. [25, Proposition 5.10])**.** *Let a, b be distinct configurations such that b can be obtained from a through a legal firing script σ. Using the monomial ordering described above, we have $x^a > x^b$.*

Proof. By assumption we have $b = a - A\sigma$. Without loss of generality, suppose σ is the firing script of a single firing at some vertex v_i. To compare the two monomials x^a and x^b, we first compare their w-weighted degrees:

$$\deg(x^a) - \deg(x^b) = w \cdot a - w \cdot b = w \cdot (A\sigma) = w^\top A\sigma \geq 0,$$

where the last inequality is given by the condition that $w^\top A \gneq \vec{\mathbf{0}}$ in Definition 1.2.4. If the inequality is strict, then we are done. If not, note that whenever v_i fires, there will be chips landing at some vertex v_j that is closer to the sink v_{n+1} than v_i is. The resulting monomial is smaller since $x_j < x_i$. $\square$

Example 2.1.14. Recall that in Example 1.2.9, we used the minimal left burning configuration $w = (6, 8, 7)$. In $R = \mathbb{C}[x_1, x_2, x_3]$, the w-weighted order on R gives us $\deg(x_1) = 6$, $\deg(x_2) = 8$, $\deg(x_3) = 7$. The graph $G_{\widetilde{A}}$ gives us $x_2 < x_1$ and $x_2 < x_3$ since only v_2 has a directed edge going into the sink. We make an arbitrary choice of $x_1 > x_3$. So if the degrees of two monomials are the same, then we use `grevlex` with the ordering of indeterminates: $x_1 > x_3 > x_2$. One can check that $x_1^5 > x_2^2x_3^2$, $x_2^5 > x_1^3x_3^3$, and $x_3^4 > x_1^2x_2^2$, where the second monomials in the three inequalities are obtained by firing at v_1, v_2, and v_3 correspondingly.

Example 2.1.15. Let B be the M-matrix defined as below, and use $w = (2, 2, 3, 2)$ and $u = (3, 4, 2, 3)$ to form $\widetilde{B}$, the (u, w)-extension of B.

$$B = \begin{pmatrix} 5 & -2 & 0 & -1 \\ -2 & 7 & -5 & -3 \\ 0 & -1 & 5 & -2 \\ -1 & -3 & -2 & 7 \end{pmatrix}, \text{ and } \widetilde{B} = \begin{pmatrix} 5 & -2 & 0 & -1 & -4 \\ -2 & 7 & -5 & -3 & -3 \\ 0 & -1 & 5 & -2 & 0 \\ -1 & -3 & -2 & 7 & -2 \\ -4 & -1 & -1 & 0 & 18 \end{pmatrix}.$$

The graph associated to $\widetilde{B}$ is shown in Figure 2.1. The w-weighted order on $R = \mathbb{C}[x_1, x_2, x_3, x_4]$ gives us $\deg(x_1) = \deg(x_2) = \deg(x_4) = 2$ and $\deg(x_3) = 3$. To order the variables, we look at the lengths of shortest paths from v_1, v_2, v_3 and v_4 to v_5 in $G_{\widetilde{B}}$. Note that there is no edge (v_4, v_5), so the shortest path from v_4 to v_5 has length 2, while it is 1 for v_1, v_2, and v_3. If we choose $x_1 > x_2 > x_3$, then we have $x_4 > x_1 > x_2 > x_3$, and one can check that, for instance, $x_4^7 > x_1x_2^3x_3^2$.

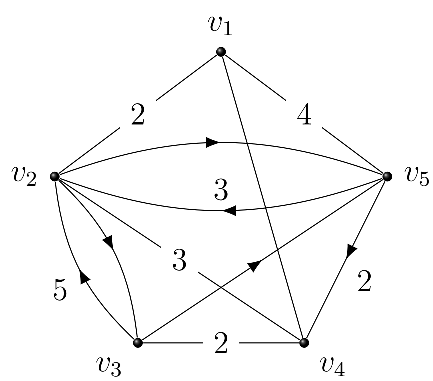

Figure 2.1: $G_{\widetilde{B}}$, the graph associated to $\widetilde{B}$.

Define $E\colon \mathbb{Z}^n \to R$ as $E(u) = x^{u^+} - x^{u^-}$ for all $u \in \mathbb{Z}^n$, and let $\mathcal{T} = E \circ A\colon \mathbb{Z}^n \to R$. Let $\{e_i\}_{i=1}^n$ denote the standard basis of $\mathbb{Z}^n$. Then $\mathcal{T}(e_i) = t_i$ is the i-th toppling polynomial.

Proposition 2.1.16 (c.f. [25, Theorem 5.11])**.** *Let $A \in M_{n\times n}(\mathbb{Z})$ be a non-singular M-matrix. Let b be a burning configuration, and let σ_b be its script. Then*

$$\Gamma_b = \{\mathcal{T}(\sigma) \mid 0 \leq \sigma \leq \sigma_b\}$$

is a Gröbner basis for $I(A)$.

Proof. By definition of $I(A)$, we have $\operatorname{im}(\mathcal{T}) \subseteq I(A)$. Also, for b any burning configuration, $\mathcal{T}(\sigma_b) = x^b - 1$. So Γ_b generates $I(A)$ by Proposition 2.1.4.

Let σ_1, σ_2 be scripts with $\sigma_1, \sigma_2 \leq \sigma_b$. For $i = 1, 2$, define $c_i = A\sigma_i$, and write

$$\mathcal{T}(\sigma_i) = x^{c_i^+} - x^{c_i^-}.$$

Observe that c_i^- is the configuration obtained from c_i^+ through firing the script σ_i. Thus by Proposition 2.1.13, $x^{c_i^+}$ is the leading term in $\mathcal{T}(\sigma_i)$ with respect to the monomial ordering we put on R. Now define

$$x^{a_i} = \frac{\operatorname{lcm}(x^{c_1^+}, x^{c_2^+})}{x^{c_i^+}}$$

for $i = 1, 2$ so that $a_1 + c_1^+ = a_2 + c_2^+ = c$ for some configuration c. It remains to show that the S-polynomial,

$$S(\mathcal{T}(\sigma_1), \mathcal{T}(\sigma_2)) = x^{a_1}\mathcal{T}(\sigma_1) - x^{a_2}\mathcal{T}(\sigma_2) = x^{a_2+c_2^-} - x^{a_1+c_1^-},$$

is reduced to 0 by Γ_b.

Define a script $\tau \in \mathbb{N}^n$ as $\tau_j = \max(\sigma_{1,j}, \sigma_{2,j})$. Since by assumption $\sigma_1, \sigma_2 \leq \sigma_b$, we have $\tau \leq \sigma_b$. Let $c' = c - A\tau$ denote the configuration obtained from c by firing τ, and let us consider a decomposition of the script τ via the firing sequence

$$a_i + c_i^+ \xrightarrow{\sigma_i} a_i + c_i^- \xrightarrow{\tau - \sigma_i} c'.$$

Also, note that

$$\mathcal{T}(\tau - \sigma_i) = \frac{\operatorname{lcm}(x^{c_1^+}, x^{c_2^+})}{x^{c_i^+}} - \frac{\operatorname{lcm}(x^{c_1^-}, x^{c_2^-})}{x^{c_i^-}}.$$

One can hence check that

$$S(\mathcal{T}(\sigma_1), \mathcal{T}(\sigma_2)) = x^{a_2+c_2^-} - x^{a_1+c_1^-} = x^{c_2^-}\mathcal{T}(\tau - \sigma_2) - x^{c_1^-}\mathcal{T}(\tau - \sigma_1).$$

□

2.2 Lattice Ideal from the Extended Matrix

In this section, we will find a Gröbner basis for $I(\widetilde{A}) \subseteq S = \mathbb{C}[x_1, \ldots, x_{n+1}]$. To do so, we need a monomial ordering on S that is *compatible* with the ordering on R. Namely, it is a $(w, 1)$-weighted `grevlex` order that extends the ordering on R and satisfies the condition $x_i > x_{n+1}$ for all $1 \leq i \leq n$. We say that $x^a x_{n+1}^d > x^b x_{n+1}^e$ if $(w, 1) \cdot (a, d) > (w, 1) \cdot (b, e)$, or if $(w, 1) \cdot (a, d) = (w, 1) \cdot (b, e)$ and $d < e$, or if $(w, 1) \cdot (a, d) = (w, 1) \cdot (b, e)$, $d = e$, and $x^a > x^b$.

Definition 2.2.1 (Homogenization)**.** We define the *degree of a polynomial* in either R or S to be the largest degree of its monomials' degrees. For $f \in R$, we can *homogenize* f to some monomial $f^h \in S$ such that each monomial in f has degree $\deg(f)$:

$$f^h \overset{\text{def}}{=} x_{n+1}^{\deg(f)} f\left(\frac{x_1}{x_{n+1}^{w_1}}, \dots, \frac{x_n}{x_{n+1}^{w_n}}\right).$$

Similarly, given $F \in S$, we define the *dehomogenization of* F to be $F(x_1, \dots, x_n, 1)$, i.e., we evaluate F at $x_{n+1} = 1$. Finally, given some ideal $I \subseteq R$, we can define the *homogenization of* I in S as

$$I^h \overset{\text{def}}{=} \langle f^h \mid f \in I \rangle.$$

Remark 2.2.2. Let $f \in R$. Note that $f^h(x_1, \dots, x_n, 1) = f$, i.e., homogenize and then dehomogenize gives back the original polynomial. On the other hand, if f is the dehomogenization of some $F \in S$, then $F = x_{n+1}^d f^h$, where d is the largest power of x_{n+1} such that F is divisible by x_{n+1}^d.

Lemma 2.2.3 ([9, 8.4 Theorem 4])**.** *For any ideal $I \subseteq R$, if Γ is a Gröbner basis of I, then $\Gamma^h = \{g^h \mid g \in \Gamma\}$ is a Gröbner basis for I^h.*

Proof. First we show that for any $f \in R$, $\mathrm{LT}(f^h) = \mathrm{LT}(f)$. Indeed, if there is only one term m in f with $\deg(m) = \deg(f)$, then all other terms in f^h are divisible by x_{n+1} and hence smaller than m even after homogenization. On the other hand, if there are two terms m, n with $\deg(m) = \deg(n) = \deg(f)$ and $m > n$, then both terms are not divisible by x_{n+1} after homogenization, and as a result we still have $m > n$.

Now let $F \in I^h$. By definition, we can write F as $F = \sum_{i=1}^k h_i f_i^h$ for some $h_i \in S$, $f_i \in I$. Let f denote the dehomogenization of F, and we have

$$f = F(x_1, \dots, x_n, 1) = \sum_{i=1}^k h_i(x_1, \dots, x_n, 1) f_i \in I.$$

Since $F = x_{n+1}^d f^h$ for some d, we also have

$$\mathrm{LT}(F) = x_{n+1}^d \,\mathrm{LT}(f^h) = x_{n+1}^d \,\mathrm{LT}(f).$$

Since Γ is a Gröbner basis for I, $\mathrm{LT}(f)$ is divisible by $\mathrm{LT}(g)$ for some $g \in \Gamma$, or otherwise f cannot be reduced to zero by Γ. So $\mathrm{LT}(F)$ is divisible by $\mathrm{LT}(g) = \mathrm{LT}(g^h)$ as well, and we may conclude that Γ^h is a Gröbner basis for I^h. □

Proposition 2.2.4 (c.f. [25, Proposition 4.8])**.** *Let $A \in M_{n\times n}(\mathbb{Z})$ be an M-matrix, and let $u, w \in \mathbb{N}^n$ be such that we can form $\widetilde{A} \in M_{(n+1)\times(n+1)}(\mathbb{Z})$, the (u, w)-extension of A. Using the weighted sandpile monomial ordering on R and a compatible ordering on S, we have $I(\widetilde{A}) = I(A)^h$.*

Proof. ($\subseteq$) This inclusion follows from construction of $\widetilde{A}$. Let

$$\widetilde{a} = \begin{pmatrix} a \\ a_{n+1} \end{pmatrix}, \widetilde{b} = \begin{pmatrix} b \\ b_{n+1} \end{pmatrix} \in \mathbb{N}^{n+1}$$

be such that $\widetilde{a} - \widetilde{b} \in \mathrm{im}_{\mathbb{Z}}(\widetilde{A})$. Then there is some $c \in \mathbb{Z}^{n+1}$ such that $\widetilde{a} - \widetilde{b} = \widetilde{A}c$. Taking the dot product of both sides with $(w, 1)$ gives:

$$(w,1) \cdot (\widetilde{a} - \widetilde{b}) = (w,1) \cdot (\widetilde{A}c) = (w,1)^\top \left(\begin{array}{c|c} A & -Au \\ \hline -w^\top A & w^\top Au \end{array} \right) c = \vec{\mathbf{0}} \cdot c = 0.$$

Therefore,

$$w \cdot (a - b) = b_{n+1} - a_{n+1}.$$

Without loss of generality, suppose $w \cdot (a - b) \geq 0$. Now observe that:

$$x_{n+1}^{a_{n+1}} (x^a - x^b)^h = x_{n+1}^{a_{n+1}} (x^a - x^b x_{n+1}^{w \cdot (a-b)}) = \widetilde{x}^{\widetilde{a}} - x^b x_{n+1}^{a_{n+1} + b_{n+1} - a_{n+1}} = \widetilde{x}^{\widetilde{a}} - \widetilde{x}^{\widetilde{b}}.$$

($\supseteq$) By Proposition 2.1.16 and Lemma 2.2.3, there is a Gröbner basis for $I(A)^h$ given by the homogenization of some generators of $I(A)$. Thus, it suffices to show the inclusion of the homogenization of the generators of $I(A)$. Let $a, b \in \mathbb{N}^n$ such that $a = b \mod \mathrm{im}_{\mathbb{Z}}(A)$, i.e., there is $c \in \mathbb{Z}^n$ such that $a - b = Ac$. If it happens to be that $w \cdot a = w \cdot b$, then $(x^a - x^b)^h = x^a - x^b \in I(\widetilde{A})$. Without loss of generality, suppose that $w \cdot a > w \cdot b$. Thus x^a has a larger total degree than x^b. Denote this difference in degree as d. Then

$$\begin{aligned} d &= w \cdot a - w \cdot b \\ &= w \cdot (a - b) \\ &= w \cdot (Ac) \\ &= w^\top Ac. \end{aligned}$$

By assumption, $(x^u - x^v)^h = x^u - x^v x_{n+1}^d$. Check that

$$\widetilde{A} \begin{pmatrix} c \\ 0 \end{pmatrix} = \begin{pmatrix} Ac \\ -w^\top Ac \end{pmatrix} = \begin{pmatrix} u - v \\ -d \end{pmatrix} = \begin{pmatrix} u \\ 0 \end{pmatrix} - \begin{pmatrix} v \\ d \end{pmatrix}.$$

It follows from definition of $I(\widetilde{A})$ that $(x^u - x^v)^h \in I(\widetilde{A})$. $\square$

Corollary 2.2.5. *Let b be a right burning configuration of A with script σ_b. Define $\Gamma_b = \{\mathcal{T}(\sigma) \mid \vec{\mathbf{0}} \lneq \sigma \leq \sigma_b\}$. Then $\Gamma^h = \{g_i^h \mid g_i \in \Gamma_b\}$ is a Gröbner basis for $I(\widetilde{A})$.*

2.3 Minimal Free Resolutions of Lattice Ideals

In order to have a better understanding of the structure of lattice ideals, we look at their minimal free resolutions. To begin with, we introduce chain complexes of free modules. For more details, see [24] and [11, Chapter 6].

We say that an R-module F is a *free module of rank r* if there is some isomorphism $F \cong R^r$ for some $r > 0$. If F is graded, usually by $\mathbb{N}^n$, then the isomorphism can be written as $F \cong \bigoplus_{i=1}^r R(-a_i)$ for some $a_i \in \mathbb{N}^n$. A sequence of maps of free R-modules

$$\mathcal{F}_\bullet : 0 \longleftarrow F_0 \xleftarrow{\phi_1} F_1 \longleftarrow \cdots \longleftarrow F_{l-1} \xleftarrow{\phi_l} F_l \longleftarrow 0 \tag{2.3.1}$$

is a *complex* if $\phi_i \circ \phi_{i-1} = 0$ for all i. The complex is *exact* in homological degree i if $\ker(\phi_i) = \mathrm{im}(\phi_{i+1})$. When the free modules F_i are graded, we require the maps ϕ_i to be degree-preserving.

Definition 2.3.2. A complex $\mathcal{F}_{\bullet}$ as in 2.3.1 is a *free resolution* of a module M over R if it is exact everywhere except at homological degree 0, where $M = F_0/\mathrm{im}(\phi_0)$. We can *augment* the free resolution $\mathcal{F}_{\bullet}$ by placing $0 \longleftarrow M \xleftarrow{\phi_0} F_0$ in the left end instead. The image of ϕ_{i+1} in F_i is the i^{th} *syzygy module* of M. The *length* of the resolution is the greatest homological degree of a nonzero module in this resolution.

The Hilbert syzygy theorem says that every module over the polynomial ring R has a free resolution with length at most n. In particular, if $M = R/I$ for some ideal $I \subseteq R$, then M has a free resolution with length at most n. We can always find a *minimal* free resolution of M, such that the ranks of the free modules F_i in the complex are minimal. The minimal free resolutions of a module are isomorphic, meaning if $\mathcal{F}_{\bullet}$ and $\mathcal{F}'_{\bullet}$ are two minimal free resolutions of M, then we have an isomorphism of free R-modules $F_i \cong F'_i$ for each i. Therefore, the ranks of free modules in the minimal free resolution of M is well-defined.

Definition 2.3.3. Let M be a finitely generated $\mathbb{N}^n$ graded R-module, and $\mathcal{F}_{\bullet}$ be a minimal free resolution of M with $F_i = \bigoplus_{a\in\mathbb{N}^n} R(-a)^{\beta_{i,a}}$. We define the *i-th Betti number of M in degree a* to be the invariant $\beta_{i,a} = \beta_{i,a}(M)$.

Definition 2.3.4. A *simplicial complex* Δ on the vertex set $[n] = \{1, 2, \ldots, n\}$ is a collection of subsets of $[n]$ called *faces* that is closed under taking subsets, i.e., if we have $\sigma \in \Delta$ and there is $\tau \subseteq \sigma \subseteq [n]$, then $\tau \in \Delta$. A face σ of cardinality $|\sigma| = i + 1$ has *dimension* i and is called an *i-face* of Δ. The dimension $\dim(\Delta)$ is the maximum of the dimensions of its faces. A *facet* of Δ is a face such that none of its superset is in Δ.

Example 2.3.5. Let $n = 4$, and define Δ to be a simplicial complex with facets $\{2, 3, 4\}, \{1, 2\}, \{1, 3\}$. We can thus draw Δ as shown below in Figure 2.2:

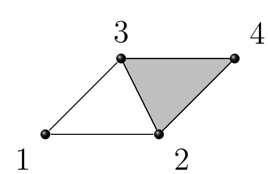

Figure 2.2: The simplicial complex Δ

Let Δ be a simplicial complex on $[n]$, and for each i we use $F_i(\Delta)$ to denote the set of i-dimensional faces of Δ. Let $\Bbbk^{F_i(\Delta)}$ be the vector space over a field $\Bbbk$ with basis elements e_σ corresponding to $\sigma \in F_i(\Delta)$. For $\sigma \in \Delta$ and $j \in \sigma$ the r-th element of σ written in increasing order, define the *sign* of j in σ as $\mathrm{sign}(j, \sigma) = (-1)^{r-1}$.

Definition 2.3.6. The (*augmented* or *reduced*) *chain complex* of Δ over $\Bbbk$ is the complex $\widetilde{\mathcal{C}}_{\cdot}(\Delta, \Bbbk)$:

$$0 \longleftarrow \Bbbk^{F_{-1}(\Delta)} \xleftarrow{\partial_0} \cdots \Bbbk^{F_{i-1}(\Delta)} \xleftarrow{\partial_i} \Bbbk^{F_i(\Delta)} \longleftarrow \cdots \xleftarrow{\partial_{n-1}} \Bbbk^{F_{n-1}(\Delta)} \longleftarrow 0,$$

where the *boundary maps* ∂_i is defined as

$$\partial_i(e_\sigma) = \sum_{j \in \sigma} \operatorname{sign}(j, \sigma) e_{\sigma \setminus \{j\}}.$$

The boundary maps can be thought of as picking an orientation of the faces, which starts from the vertex with the smallest label, and traverses through the vertices with increasing labels.

Example 2.3.7. Let us return to Δ in Example 2.3.5. Fix an orientation as in Figure 2.3, and the boundary maps over $\mathbb{C}$ can be written as:

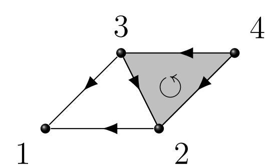

Figure 2.3: Δ with an orientation

The chain complex $\mathcal{C}_{\cdot}(\Delta, \mathbb{C})$ can be written out as:

$$0 \xleftarrow[0]{} \mathbb{C} \longleftarrow \mathbb{C}^4 \longleftarrow \mathbb{C}^5 \longleftarrow \mathbb{C} \xleftarrow[0]{} 0$$

$$\begin{array}{c} \\ \emptyset \end{array} \begin{array}{cccc} 1 & 2 & 3 & 4 \\ \end{array} \qquad \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline \emptyset & 1 & 1 & 1 & 1 \end{array}$$

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 12 & 13 & 23 & 24 & 34 \end{array} \\ \begin{bmatrix} -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array} \qquad \begin{array}{c} \\ 12 \\ 13 \\ 23 \\ 24 \\ 34 \end{array} \begin{array}{c} 234 \\ \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \end{array}$$

A *monomial ideal* is an ideal that is generated by monomials. We can form its *Scarf complex* to compute a minimal free resolution of a monomial ideal.

Definition 2.3.8 (Scarf Complex)**.** Let $M \subseteq R$ be a monomial ideal with a minimal generating set $x^{a_1}, \ldots, x^{a_r}$. For $\sigma \subseteq [r]$, we define $a_\sigma \in \mathbb{N}^n$ as

$$(a_\sigma)_j = \max(\{(a_i)_j \mid i \in \sigma\}).$$

The *Scarf complex* of M, denoted Δ_M, is defined to be

$$\Delta_M = \{\sigma \subseteq [r] \mid a_\sigma = a_\tau \implies \sigma = \tau\}.$$

Namely, if $\sigma \in \Delta_M$, then $x^{a_\sigma} = \operatorname{lcm}(\{x^{a_i} \mid i \in \sigma\}) \stackrel{\text{call}}{=} s_\sigma$ is the unique least common multiple of all of the generators $x^{a_1}, \ldots, x^{a_r}$.

Lemma 2.3.9 ([11, Lemma 6.8])**.** *The Scarf complex Δ_M is a simplicial complex. Its dimension is at most $n-1$.*

Definition 2.3.10. The *algebraic Scarf complex* of M, denoted $\mathcal{F}_M$, is the augmented chain complex of Δ_M over R, that is, at each level, we have $F_i = R^{F_{i-1}(\Delta_M)}$, the free module over R with basis elements being the $(i-1)$-faces of Δ_M. The boundary maps are written as

$$\partial_i(e_\sigma) = \sum_{j\in\sigma} \operatorname{sign}(j,\sigma)\frac{s_\sigma}{s_{\sigma\setminus\{j\}}} e_{\sigma\setminus\{j\}},$$

and the faces are labeled as s_σ instead of σ.

Definition 2.3.11. A monomial s' *strictly divides* another monomial s if s' divides s/x_i for all variables x_i dividing s. A monomial ideal $\langle s_1,\ldots,s_r\rangle$ is *generic* if whenever two distinct minimal generators s_i and s_j have the same positive (nonzero) degree in some variable, a third generator s_k strictly divides their least common multiple $\operatorname{lcm}(s_i,s_j)$.

Example 2.3.12. For example, x_1x_2 strictly divides $x_1^2x_2^2x_3$, but it does not strictly divide $x_1x_2^2x_3$. The monomial ideal $I = \langle x_1x_2, x_1^2, x_2^2\rangle$ is generic since the condition is vacuously satisfied. The monomial ideal $J = \langle x_1x_2x_3, x_1^2x_3, x_1^3x_2\rangle$ is not generic, since $x_1^2x_3$ does not strictly divide $\operatorname{lcm}(x_1^3x_2, x_1x_2x_3) = x_1^3x_2x_3$. Now we draw Δ_I and Δ_J, the Scarf complexes for I and J, in Figure 2.4, with the labels of the vertices replaced by the corresponding monomial generators:

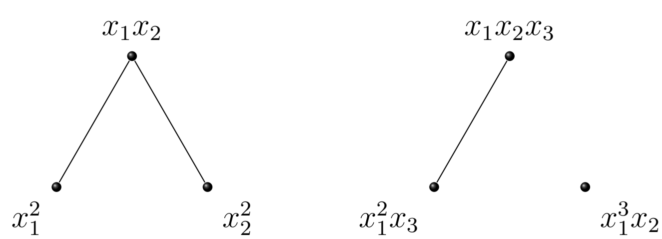

Figure 2.4: The Scarf complexes for I and J.

Theorem 2.3.13 ([11, Proposition 6.12, Theorem 6.13])**.** *If M is a monomial ideal in R, then every free resolution of R/M contains the algebraic Scarf complex $\mathcal{F}_M$ as a subcomplex. If M is a generic monomial ideal, then $\mathcal{F}_M$ minimally resolves the quotient R/M.*

Now we define the Scarf complex of a lattice ideal, using a construction similar to the Scarf complex of a monomial ideal.

Let $\mathcal{L}$ be a sublattice of $\mathbb{Z}^n$ satisfying $\mathcal{L}\cap\mathbb{N}^n = \{\vec{\mathbf{0}}\}$, i.e., $\mathcal{L}$ contains no nonnegative vectors. This is to ensure that the lattice ideal $I_{\mathcal{L}}$ is homogeneous with respect to some weight vector $w \geq \vec{\mathbf{1}}$. For any finite subset $J \subseteq \mathcal{L}$, we define $\max(J) \in \mathbb{Z}^n$ to be the vector which is the coordinate-wise maximum of J, meaning

$$(\max(J))_i = \max(\{a_i \mid a \in J\}).$$

We define an infinite simplicial complex $\Delta_{\mathcal{L}}$ as:

$$\Delta_{\mathcal{L}} = \{J \subseteq \mathcal{L} \mid \max(J) \neq \max(J') \text{ for all finite } J' \subseteq L \text{ other than } J\}.$$

This is an infinite simplicial complex of dimension of at most $n-1$. There is a natural transitive action of $\mathcal{L}$ on $\Delta_{\mathcal{L}}$ given by translation, i.e., for any $a \in \mathcal{L}$, $J \in \Delta_{\mathcal{L}}$ if and only if $J + a \in \Delta_{\mathcal{L}}$.

Definition 2.3.14. Following the notation as above, for a lattice $\mathcal{L} \subseteq \mathbb{Z}^n$ with no nonnegative vectors, we identify $\Delta_{\mathcal{L}}$ with its poset of nonempty faces, and define the *Scarf complex* of $\mathcal{L}$ to be the quotient poset $\Delta_{\mathcal{L}}/\mathcal{L}$.

Remark 2.3.15. For $a \in \mathcal{L}$, the *link of* a in the complex $\Delta_{\mathcal{L}}$ is the subcomplex

$$\Delta_{\mathcal{L}}^a \overset{\text{def}}{=} \{J \subseteq L \backslash \{a\} \mid J \cup \{a\} \in \Delta_{\mathcal{L}}\}.$$

We may identify the Scarf complex $\Delta_{\mathcal{L}}/\mathcal{L}$ with $\Delta_{\mathcal{L}}^{\vec{\mathbf{0}}}$ modulo the action by $\mathcal{L}$. It turns out that $\Delta_{\mathcal{L}}$ is locally finite, meaning $\Delta_{\mathcal{L}}^a$ is finite for all $a \in \mathcal{L}$, and that the Scarf complex $\Delta_{\mathcal{L}}/\mathcal{L}$ is finite (see [24, Section 2]).

Example 2.3.16. Let $\mathcal{L} \subseteq \mathbb{Z}^2$ be the lattice spanned by $(1, -1)$. The complex $\Delta_{\mathcal{L}}$ has facets formed by the consecutive pairs $\{(i, -i), (i + 1, -i - 1)\}_{i \in \mathbb{Z}}$. Its Scarf complex $\Delta_{\mathcal{L}}/\mathcal{L}$ consists of one face of dimension 1 and one of dimension 0, since all vertices are identified to be the same as $(0, 0)$ under the action by $\mathcal{L}$.

Remark 2.3.17. Let $A \in M_{n \times n}(\mathbb{Z})$ be a nonsingular M-matrix, and $\widetilde{A}$ be its (u, w)-extension. We always have that $\operatorname{im}_{\mathbb{Z}}(\widetilde{A}) \cap \mathbb{N}^{n+1} = \{\vec{\mathbf{0}}\}$. Let $v \in \mathbb{Z}^n$. We will show that there is no $k \in \mathbb{Z}$ such that $\widetilde{A} \cdot (v, k) > \vec{\mathbf{0}}$. Consider the multiplication:

$$\widetilde{A}\begin{pmatrix} v \\ k \end{pmatrix} = \left(\begin{array}{c|c} A & -Au \\ \hline -w^\top A & w^\top Au \end{array}\right)\begin{pmatrix} v \\ k \end{pmatrix} = \begin{pmatrix} Av - kAu \\ -w^\top Av + kw^\top Au \end{pmatrix} = \begin{pmatrix} A(v - ku) \\ -w^\top A(v - ku) \end{pmatrix}.$$

Since $w \geq \vec{\mathbf{1}}$, we have that $A(v-ku) \geq \vec{\mathbf{0}}$ if and only if $-w^\top A(v-ku) \leq 0$. Therefore, $\widetilde{A}(v, k) \geq \vec{\mathbf{0}}$ implies that $\widetilde{A}(v, k) = \vec{\mathbf{0}}$.

Definition 2.3.18. We first identify the Scarf complex $\Delta_{\mathcal{L}}/\mathcal{L}$ with $\Delta_{\mathcal{L}}^{\vec{\mathbf{0}}}$ modulo the action by $\mathcal{L}$. For $J \in \Delta_{\mathcal{L}}/\mathcal{L}$, let C_J be the set of monomials

$$C_J = \{x^{\max(J)-a} \mid a \in J\}.$$

The *algebraic Scarf complex* of the lattice ideal $I_{\mathcal{L}}$ is defined to be the complex of free R-modules

$$\mathcal{F}_{\mathcal{L}} = \bigoplus_{J \in \Delta_{\mathcal{L}}/\mathcal{L}} R \cdot E_J,$$

where E_J denotes a basis vector in homological degree $|J| - 1$, and the sum runs through all faces of $\Delta_{\mathcal{L}}/\mathcal{L}$. The differential map ∂ is defined as

$$\partial(E_J) = \sum_{a \in J} \operatorname{sign}(a, J) \cdot \gcd(C_{J \backslash \{a\}}) \cdot E_{J \backslash \{a\}},$$

where we define $\operatorname{sign}(a, J) = (-1)^{r-1}$ if $x^{\max(J)-a}$ is in the r-th position under the lexicographic ordering of C_J.

Remark 2.3.19. The set of monomials C_J in Definition 2.3.18 is a *basic fiber* of $\mathcal{L}$. The set of all monomials of a fixed degree in $\mathbb{Z}^n/\mathcal{L}$ is called a *fiber*. A fiber C is *basic* if $\gcd(C) = 1$ and $\gcd(C\backslash\{m\}) \neq 1$ for all $m \in C$. For any $m \in C$, the monomials in $C\backslash\{m\}$ divided by their greatest common divisor form a basic fiber. So we have a poset structure on the set of all basic fibers of $\mathcal{L}$. It turns out that this poset is isomorphic to the Scarf complex $\Delta_{\mathcal{L}}/\mathcal{L}$. See [24, Section 2] for more details.

Definition 2.3.20. A lattice ideal $I_{\mathcal{L}}$ is *generic* if it is generated by binomials of full support, i.e., if $x^u - x^v$ is a generator, then $\operatorname{supp}(u-v) = [n]$.

Remark 2.3.21. Genericity of lattice ideal does not guarantee the initial ideal being *strongly generic*, meaning there is no generator having the same nonzero degree in some variable [24, Example 4.5]. Whether genericity of lattice ideal implies genericity of its initial ideal is still a question to be answered.

Theorem 2.3.22 ([24, Theorem 4.2])**.** *The algebraic Scarf complex $\mathcal{F}_{\mathcal{L}}$ is contained in the minimal free resolution of $R/I_{\mathcal{L}}$. If $I_{\mathcal{L}}$ is generic, then $\mathcal{F}_{\mathcal{L}}$ is the minimal free resolution of $R/I_{\mathcal{L}}$.*

Fix some monomial ordering on R. We compute the algebraic Scarf complex of a generic lattice ideal through the algebraic Scarf complex of its initial ideal, which is a finite simplicial complex and is much easier to compute.

We use $x^{a_i^+} - x^{a_i^-}$ to denote the generators of the lattice ideal $I_{\mathcal{L}}$ and assume that x_n is the smallest variable and that it divides $x^{a_i^-}$.

Theorem 2.3.23 ([24, Theorems 5.2, 5.4])**.** *Let $I_{\mathcal{L}}$ be a generic lattice ideal. Fix a monomial ordering, and let $M = \operatorname{In}(I_{\mathcal{L}})$. Use $\Delta_{\mathcal{L}}/\mathcal{L}$ and Δ_M to denote the Scarf complexes of $I_{\mathcal{L}}$ and M correspondingly. Then*

1. *The i-faces of Δ_M are in bijection with the $(i+1)$-faces of $\Delta_{\mathcal{L}}/\mathcal{L}$.*

2. *The poset $\Delta_{\mathcal{L}}/\mathcal{L}$ is derived from Δ_M in the following way. Let J be a face of Δ_M and hence of $\Delta_{\mathcal{L}}/\mathcal{L}$. Then J covers $J\backslash\{j\}$ for all $j \in J$, and it covers one additional face $\widetilde{J}$ as follows: set $m_J = \operatorname{lcm}(\{x^{a_j^+} \mid j \in J\})$, consider the set of monomials $\{\frac{m_J}{x^{a_j^+}} x^{a_j^-} \mid j \in J\}$ divided by their greatest common divisors, and let p be the unique monomial among them which is not divisible by x_n; then $\widetilde{J}$ is the unique $(|J|-1)$-face of Δ_M such that $\operatorname{lcm}(\{x^{a_j^+} \mid j \in \widetilde{J}\}) = p$.*

Corollary 2.3.24 ([24, Corollary 5.5])**.** *The minimal free resolution of $R/I_{\mathcal{L}}$ is the free R-module*

$$\mathcal{F}_{\mathcal{L}} = \bigoplus_{J \in \Delta_M} R \cdot E_J,$$

where the basis element E_J is placed in homological degree $|J|$. The differential map is defined as

$$\partial(E_J) = \sum_{i \in J} \operatorname{sign}(i, J) \cdot \frac{m_J}{\operatorname{lcm}(\{x^{a_i^+} \mid j \in J\backslash\{i\}\})} \cdot E_{J\backslash\{i\}} \pm \widetilde{m}_J \cdot E_J,$$

where $m_J = \mathrm{lcm}(\{x^{a_j^+} \mid j \in J\})$, $\widetilde{m}_J = \gcd(\{x^{a_j^-} \mid j \in J\})$, *and* $\widetilde{J}$ *is described in Theorem 2.3.23. The sign of the last term is determined by the condition* $\partial^2 = 0$.

Remark 2.3.25. Note that if we erase the last term of the boundary maps, we get the algebraic Scarf complex of the initial ideal, $\mathcal{F}_M$.

Chapter 3

Cyclic Groups

3.1 Faithful Representations of the Cyclic Group

Let $C_n = \langle g \mid g^n = e \rangle$ denote the cyclic group of n elements with generator g. All its irreducible representations over $\mathbb{C}$ are of dimension 1, and they are distinguished from each other by sending g to different n-th roots of unity. Put $\zeta = e^{2\pi i/n}$, a primitive n-th root of unity, and let ρ_i denote the representation of C_n given by $g \mapsto \zeta^i$, where $1 \leq i \leq n$.

Let $\rho\colon C_n \to \mathrm{GL}_m(\mathbb{C})$ be a representation of dimension m. We can decompose it into a direct sum of these irreducible representations, $\rho \cong \bigoplus_{i=1}^n m_i \rho_i$ for some $m_i \in \mathbb{N}$ such that $\sum_{i=1}^n m_i = m$. Now we want to form the extended McKay-Cartan matrix $\widetilde{C}$ for this representation as described in Section 1.3. For $1 \leq k \leq n$, one may check that

$$\rho \otimes \rho_k \cong \left(\bigoplus_{i=1}^n m_i \rho_i \right) \otimes \rho_k \cong \bigoplus_{i=1}^n m_i (\rho_i \otimes \rho_k) \cong \bigoplus_i m_i \rho_{i+k}.$$

The subscripts $i + k$ live in $\mathbb{Z}/n\mathbb{Z}$. So we form the matrix M with $M_{i,j} = m_{j-i}$ and the extended McKay-Cartan matrix $\widetilde{C}$:

$$M = \begin{pmatrix} m_n & m_1 & \dots & m_{n-2} & m_{n-1} \\ m_{n-1} & m_n & \dots & m_{n-3} & m_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ m_2 & m_3 & \dots & m_n & m_1 \\ m_1 & m_2 & \dots & m_{n-1} & m_n \end{pmatrix},$$

$$\widetilde{C} = mI_m - M^\top = \begin{pmatrix} m - m_n & -m_{n-1} & \dots & -m_2 & -m_1 \\ -m_1 & m - m_n & \dots & -m_3 & -m_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -m_{n-2} & -m_{n-3} & \dots & m - m_n & -m_{n-1} \\ -m_{n-1} & -m_{n-2} & \dots & -m_1 & m - m_n \end{pmatrix}.$$

The McKay-Cartan matrix is:

$$C = \begin{pmatrix} m - m_n & -m_{n-1} & \dots & -m_2 \\ -m_1 & m - m_n & \dots & -m_3 \\ \vdots & \vdots & \ddots & \vdots \\ -m_{n-2} & -m_{n-3} & \dots & m - m_n \end{pmatrix}.$$

The representation ρ is faithful when it has trivial kernel, i.e., $\ker(\rho) = \{e\}$. Amongst the irreducible representations, ρ_i is faithful when $\gcd(i, n) = 1$. Therefore, $\rho \cong \bigoplus_{i=1}^{n} m_i \rho_i$ is faithful only when $\gcd(\{i \mid m_i \neq 0\} \cup \{n\}) = 1$. By Theorem 1.3.2, in order to study the lattice ideal $I(\widetilde{C})$, we restrict ourselves to such kinds of representations of C_n.

Observe that the extended McKay-Cartan matrix $\widetilde{C}$ is the full Laplacian of a directed graph G, and that the McKay-Cartan matrix C is the reduced Laplacian of G with respect to the vertex corresponding to the trivial representation. Using properties of the cyclic group, we may describe G in the following way:

For $1 \leq i < n$, edges of weight m_i start at v_j and end at v_{j+i} for $1 \leq j \leq n$. One can see that if $\gcd(i, n) = 1$, then the subgraph of G containing only edges of weight m_i is a directed cycle that starts from v_i and passes through all vertices.

For simplicity, we only consider the case when $m_i \neq 0$. Using results from earlier, we can describe the generators of the lattice ideal $I(C)$ and give Gröbner bases for $I(C)$ and for $I(\widetilde{C})$.

By Proposition 1.3.3, $\vec{\mathbf{1}}$ is a burning script of C. Since $\vec{\mathbf{1}}C > \vec{\mathbf{0}}$, there is a directed edge from all vertices to the sink v_n in $G_{\widetilde{C}}$. So the $\vec{\mathbf{1}}$-weighted sandpile monomial ordering on $\mathbb{C}[x_1, \dots, x_{n-1}]$, introduced in Section 2.1, is any `grevlex` ordering. The monomial ordering on $R = \mathbb{C}[x_1, \dots, x_n]$ extending this ordering is still any `grevlex` ordering satisfying the condition that x_n is the smallest indeterminate. Therefore, we will use the normal `grevlex` ordering, where we say $x^a > x^b$ if $\vec{\mathbf{1}} \cdot a > \vec{\mathbf{1}} \cdot b$ or if $\vec{\mathbf{1}} \cdot a = \vec{\mathbf{1}} \cdot b$ and the rightmost nonzero entry of $a - b$ is negative.

By Proposition 2.1.4, the lattice ideal $I(C) \subseteq \mathbb{C}[x_1, \dots, x_{n-1}]$ is generated by the toppling polynomials $\{\prod_{i=1}^{n-1} x_i^{(c_i^+)} - \prod_{i=1}^{n-1} x_i^{(c_i^-)} \mid c \text{ is a column of } C\}$ along with the binomial $\prod_{i=1}^{n-1} x_i^{m_i} - 1$. By Proposition 2.1.16, a Gröbner basis Γ for $I(C)$ is given by

$$\Gamma = \{\textstyle\prod_{i=1}^{n-1} x_i^{(C\sigma)_i^+} - \prod_{i=1}^{n-1} x_i^{(C\sigma)_i^-} \mid \vec{\mathbf{0}} \lneq \sigma \leq \vec{\mathbf{1}}\}, \tag{3.1.1}$$

i.e., binomials corresponding to sums of columns of C.

Note that the positive term in each binomial is always the leading term by Proposition 2.1.13. Therefore, in the homogenization Γ^h, the indeterminate x_n only appears in the negative terms of the binomials. By Corollary 2.2.5, the homogenization Γ^h is a Gröbner basis for $I(\widetilde{C})$. Thus, the initial ideal $M = \mathrm{In}(I(\widetilde{C}))$ has generators

$$M = \langle x^{(\widetilde{C}\sigma)^+} \mid \vec{\mathbf{0}} \lneq \sigma \leq \vec{\mathbf{1}} \rangle,$$

which is a minimal generating set as none of the monomials divides each other. Hence, we conclude that Γ^h is a minimal generating set of $I(\widetilde{C})$.

Using this information and the technique introduced in Section 2.3, we can describe the minimal free resolution of $I(\widetilde{C})$ by first describing the minimal free resolution of the initial ideal of $I(\widetilde{C})$. But before proceeding, let us first see an example of such a resolution when $n = 3$.

3.2 Example: the Cyclic Group with 3 Elements

In this section we work with C_3. Let $\rho\colon C_3 \to \mathrm{GL}_m(\mathbb{C})$ be a faithful representation. We decompose ρ into $\rho \cong \bigoplus_{i=1}^{3} m_i\rho_i$. First suppose that $m_i \neq 0$. The extended McKay-Cartan matrix and the McKay-Cartan matrix for ρ are listed below:

$$\widetilde{C} = \begin{pmatrix} m_1+m_2 & -m_2 & -m_1 \\ -m_1 & m_1+m_2 & -m_2 \\ -m_2 & -m_1 & m_1+m_2 \end{pmatrix}, \qquad C = \begin{pmatrix} m_1+m_2 & -m_2 \\ -m_1 & m_1+m_2 \end{pmatrix}.$$

Using the burning script $\vec{\mathbf{1}}$, we can find a generating set for the lattice ideal $I(C)$:

$$I(C) = \langle x_1^{m_1+m_2} - x_2^{m_1}, x_2^{m_1+m_2} - x_1^{m_2}, x_1^{m_1}x_2^{m_2} - 1\rangle.$$

We fix an ordering where $x_1 > x_2$. This generating set is already a Gröbner basis for $I(C)$. Thus, we can write the lattice ideal $I(\widetilde{C}) = I$ as

$$I = I(\widetilde{C}) = \langle x_1^{m_1+m_2} - x_2^{m_1}x_3^{m_2}, x_2^{m_1+m_2} - x_1^{m_2}x_3^{m_1}, x_1^{m_1}x_2^{m_2} - x_3^{m_1+m_2}\rangle,$$

and the initial ideal $\mathrm{In}(I) = M$ as

$$M = \mathrm{In}(I) = \langle x_1^{m_1+m_2}, x_2^{m_1+m_2}, x_1^{m_1}x_2^{m_2}\rangle.$$

Note that both I and M are generic. The Scarf complex of M is easy to find by hand, and we apply the method in Theorem 2.3.23 and Corollary 2.3.24 to find the minimal free resolution of R/I.

Let $\sigma_1 = (1,0), \sigma_2 = (0,1), \sigma_3 = (1,1)$. The generators are of the form $x^{(C\sigma_i)^+}$. The facets of Δ_M are $\{\sigma_1, \sigma_3\}$ and $\{\sigma_2, \sigma_3\}$. The Scarf complex Δ_M is drawn in Figure 3.1.

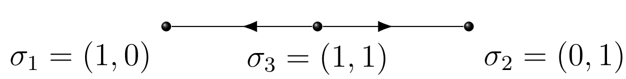

Figure 3.1: The Scarf complex of M.

We give an example of how the combinatorial rule described in Theorem 2.3.23 is applied to compute the resolution. Consider the facet $J = \{\sigma_1, \sigma_3\}$. Let $a_i = C\sigma_i$ for $\sigma_i \in J$. What is the additional face $J' \in \Delta_M$ which J covers once we move to the Scarf complex of I? Following the notation in Theorem 2.3.23, we have

$$m_J = \mathrm{lcm}(\{x^{a_i^+} \mid \sigma_i \in J\}) = x_1^{m_1+m_2}x_2^{m_2}.$$

We also have

$$\frac{m_J}{x^{a_1^+}} x^{a_1^-} = \frac{m_J}{x_1^{m_1+m_2}} x_2^{m_1} x_3^{m_2} = x_2^{m_1+m_2} x_3^{m_2},$$

$$\frac{m_J}{x^{a_3^+}} x^{a_3^-} = \frac{m_J}{x_1^{m_1} x_2^{m_2}} x_3^{m_1+m_2} = x_1^{m_2} x_3^{m_1+m_2}.$$

Divide both monomials by their greatest common divisor $x_3^{m_2}$, and we get $\{x_2^{m_1+m_2}, x_1^{m_2} x_3^{m_1}\}$. The unique monomial not divisible by x_3 is $x_2^{m_1+m_2} = p$. The only J' that satisfies the condition $\operatorname{lcm}(\{x^{a_i^+} \mid \sigma_i \in J'\}) = p$ is $J' = \{\sigma_2\}$.

Finally, Theorem 3.3.7 produces the minimal free resolution of R/I:

$$0 \longleftarrow R \xleftarrow{\partial_1} R^3 \xleftarrow{\partial_2} R^2 \longleftarrow 0,$$

where the matrices representing ∂_1 and ∂_2 are:

$$\partial_1 = \begin{array}{cc} & \begin{array}{ccc} (1,0) & (0,1) & (1,1) \end{array} \\ \emptyset & \begin{bmatrix} x_1^{m_1+m_2} - x_2^{m_1} x_3^{m_2} & x_2^{m_1+m_2} - x_1^{m_2} x_3^{m_1} & x_1^{m_1} x_2^{m_2} - x_3^{m_1+m_2} \end{bmatrix} \end{array},$$

$$\partial_2 = \begin{array}{cc} & \begin{array}{cc} \{(1,0),(1,1)\} & \{(0,1),(1,1)\} \end{array} \\ \begin{array}{c} (1,0) \\ (0,1) \\ (1,1) \end{array} & \begin{bmatrix} -x_2^{m_2} & -x_3^{m_1} \\ -x_3^{m_2} & -x_1^{m_1} \\ x_1^{m_2} & x_2^{m_1} \end{bmatrix} \end{array}.$$

3.3 Resolution of Saturated Graphs

We study the case when $m_i \neq 0$ for all i, in which case the McKay quiver of ρ is *saturated* in the sense that there is an edge from v_i to v_j for all $i \neq j$. Following [27] and [21], we will see that the Scarf complex of the initial ideal is the barycentric subdivision of the $(n-2)$-simplex. Applying Theorem 2.3.23 and Corollary 2.3.24, we obtain a minimal free resolution of the lattice ideal $I(\widetilde{C})$ and we conclude that its k-th Betti number is $(k-1)!S_{n,k}$, where $S_{n,k}$ is the Stirling number of the second kind.

Definition 3.3.1. Let $I, J \subseteq [n]$ be disjoint subsets. Define the monomial $x^{I\to J}$ as

$$x^{I\to J} = \prod_{i\in I} x_i^{-\sum_{j\in J} \widetilde{C}_{i,j}},$$

where the exponent on each x_i is the total number of edges from the vertex set $\{v_j\}_{j\in J}$ to v_i.

Remark 3.3.2. Let us see some properties of these monomials $x^{I\to J}$ that will be useful later. For disjoint I_1, I_2, J:

$$x^{I_1\cup I_2\to J} = x^{I_1\to J} x^{I_2\to J}.$$

For $J_1 \subseteq J_2$ and I disjoint from J_1 and J_2:

$$\mathrm{lcm}(x^{I\to J_1}, x^{I\to J_2}) = x^{I\to J_2},\ \gcd(x^{I\to J_1}, x^{I\to J_2}) = x^{I\to J_1}.$$

For $I_1 \subseteq I_2$ and J disjoint from I_1 and I_2:

$$\mathrm{lcm}(x^{I_1\to J}, x^{I_2\to J}) = x^{I_2\to J},\ \gcd(x^{I_1\to J}, x^{I_2\to J}) = x^{I_1\to J}.$$

One can check that for the lattice ideal $I = I(\widetilde{C})$, the homogenization Γ^h of the Gröbner basis Γ we found using the burning script $\vec{\mathbf{1}}$ (Equation (3.1.1)) is exactly the set of binomials $\{x^{I\to J} - x^{J\to I} \mid I \cup J = [n], I \cap J = \emptyset, n \in J\}$. Moreover, the initial ideal $M = \mathrm{In}(I(\widetilde{C}))$ is minimally generated by the set of monomials $\{x^{I\to[n]\setminus I} \mid I \subseteq [n-1], I \neq \emptyset\}$. Thus, Γ^h is a minimal generating set of I. Both I and M are generic, so Theorem 2.3.23 applies. We will construct a complex of free R-modules, and show that it is the algebraic Scarf complex of I.

Let $\mathrm{Cyc}_{n,k}$ denote the set of cyclically ordered partitions of the set $[n]$ into k nonempty blocks. Partitions in $\mathrm{Cyc}_{n,k}$ have the form $(I_1, \ldots, I_k)$ satisfying the conditions: $I_i \neq \emptyset$, $\bigcup_{i=1}^k I_i = [n]$, and $I_i \cap I_j = \emptyset$ for all $i \neq j$. We always use the representative where n is in the last block, i.e., $n \in I_k$. We write $R^{\mathrm{Cyc}_{n,k}}$ for the free R-module generated by these partitions. The rank of this free module is the same as the number of cyclically ordered partitions. So we have

$$|\mathrm{Cyc}_{n,k}| = (k-1)!S_{n,k},$$

where $S_{n,k}$ is the *Stirling number of the second kind*, i.e., the number of ways to partition n into k parts.

Definition 3.3.3. The $(n-1)$-*simplex*, denoted Δ^{n-1}, is a simplicial complex on $[n]$ with its only facet being $[n]$. Geometrically, it is the set

$$\Delta^{n-1} \stackrel{\text{def}}{=} \{(a_1, \ldots, a_n) \mid 0 \leq a_i \leq 1, \textstyle\sum_{i=1}^n a_i = 1 \subseteq \mathbb{R}^n\},$$

which has vertices being the standard basis $\{e_i\}$ of $\mathbb{R}^n$.

The *barycentric subdivision of* Δ^{n-1}, denoted $\mathrm{Bary}(\Delta^{n-1})$, is the simplicial complex on $V = [2^n - 1]$, which we can identify with $\{\sigma \in \mathbb{N}^n \mid \vec{\mathbf{0}} \lneq \sigma \leq \vec{\mathbf{1}}\}$ by converting $a \in [2^n - 1]$ using binary digits. Thus, we will use the n-tuples as the vertex set of $\mathrm{Bary}(\Delta^{n-1})$. For $\sigma \in V$, define

$$|\sigma| \stackrel{\text{def}}{=} |\{i \mid \sigma_i \neq 0\}|,$$

the number of nonzero entries of σ. The facets of $\mathrm{Bary}(\Delta^{n-1})$ are all of dimension $n-1$ and can be characterized using the following two properties:

1. If F is a facet of $\mathrm{Bary}(\Delta^{n-1})$, then for each $1 \leq i \leq n$, there is exactly one $\sigma \in F$ such that $|\sigma| = i$.

2. If there are $\sigma, \tau \in F$ such that $|\sigma| = |\tau| + 1$, then $\sigma - \tau = e_i$ for some i.

Geometrically, this is the same as adding a vertex to the barycenter of each face of the $(n-1)$-simplex and connecting this vertex to all vertices of each face. (The original faces of Δ^{n-1} are no longer present, of course.)

Given a $(k-1)$-face $F \in \mathrm{Bary}(\Delta^{n-1})$, we can order the vertices in F as $\sigma_1, \dots, \sigma_k$ such that $|\sigma_1| < \cdots < |\sigma_k|$. For $\sigma_i \in F$, we define the *sign of σ_i in F* to be $\mathrm{sign}(\sigma_i, F) = (-1)^{i-1}$.

Example 3.3.4. Let $n = 3$. The 2-simplex Δ^2 is the triangle with vertices labeled by $\{(0,0,1),(0,1,0),(1,0,0)\}$. Its barycentric subdivision $\mathrm{Bary}(\Delta^2)$ is the triangle but with each vertex connected to the midpoint of its opposite edge. The vertices of $\mathrm{Bary}(\Delta^2)$ are labeled by $\{(0,0,1),(0,1,0),(0,1,1),(1,0,0),(1,0,1),(1,1,0),(1,1,1)\}$, corresponding to $\{1,2,3,4,5,6,7\} = [2^3-1]$. The two simplicial complexes are drawn in Figure 3.2.

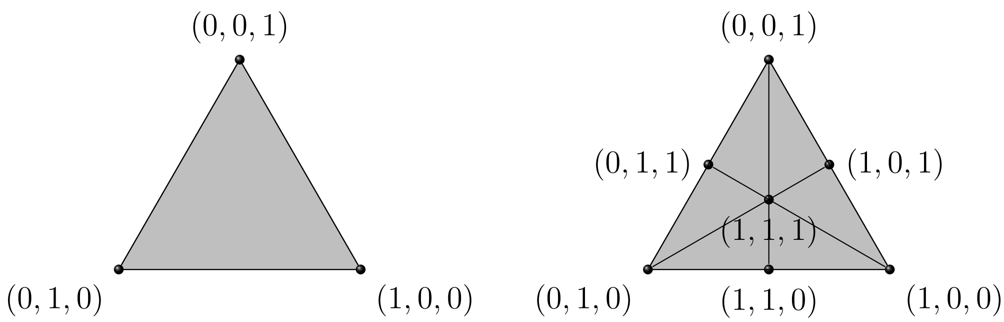

Figure 3.2: The 2-simplex Δ^2 and its barycentric subdivision $\mathrm{Bary}(\Delta^2)$.

Lemma 3.3.5. *We can impose a partial order on* $\mathrm{Cyc} = \bigcup_{k=1}^{n} \mathrm{Cyc}_{n,k}$. *Using this partial order, there is a poset isomorphism* $\mathrm{Cyc} \cong \mathrm{Bary}(\Delta^{n-2})$.

Proof. Given two partitions $I = (I_1, \dots, I_r)$, $J = (J_1, \dots, J_s)$, we say I covers J if there is some i such that for $1 \le j \le s$,

$$J_j = \begin{cases} I_j & \text{if } j < i; \\ I_j \cup I_{j+1} & \text{if } j = i; \\ I_{j+1} & \text{if } j > i. \end{cases}$$

That is, we have $s = r-1$ and J can be obtained from I by merging the two blocks I_i and I_{i+1}. The partial order on Cyc is generated by this covering relation. One can check that it is reflexive, antisymmetric, and transitive.

Now we show that there is a poset isomorphism $\mathrm{Cyc} \cong \mathrm{Bary}(\Delta^{n-2})$. First note that there is a bijection between nonempty subsets of $[n-1]$ and vertices of $\mathrm{Bary}(\Delta^{n-2})$ given by $S \subseteq [n-1]$ corresponding to $\vec{\mathbf{0}} \lneq \sigma \le \vec{\mathbf{1}}$ with $\sigma_i = 1 \iff i \in S$.

Consider $(I_1, \dots, I_k) \in \mathrm{Cyc}_{n,k}$. We can find a $(k-2)$-face in $\mathrm{Bary}(\Delta^{n-2})$ with vertices corresponding to the sets $I_1, I_1 \cup I_2, \dots, \bigcup_{j=1}^{k-1} I_j$. This is a face since all the I_j's are disjoint and nonempty. On the other hand, given $F \in \mathrm{Bary}(\Delta^{n-2})$ a $(k-2)$-face, we can find a partition in $\mathrm{Cyc}_{n,k}$ as follows: Order vertices in F as $\sigma_1, \dots, \sigma_{k-1}$ such that $|\sigma_j| < |\sigma_{j+1}|$ for all j. Define $I_1 = \{i \in [n-1] \mid (\sigma_1)_i \neq 0\}$. For $1 < j < k$, define $I_j = \{i \in [n-1] \mid (\sigma_j - \sigma_{j-1})_i \neq 0\}$. Finally, define $I_k = [n] \backslash (\bigcup_{j=1}^{k-1} I_j)$. The I_j's are nonempty and disjoint since F is a face. Also $n \in I_k$ by definition. Thus, $(I_1, \dots, I_k)$ is a partition in $\mathrm{Cyc}_{n,k}$.

It remains to show that the covering relation is preserved. Consider the partition $I = (I_1, \dots, I_r)$ and its corresponding face $F = \{\sigma_1, \dots, \sigma_{r-1}\}$, where σ_k is the tuple corresponding to the set $\bigcup_{j=1}^{k} I_j$. Choose some $i < r$ and form the partition $I' = (I_1, \dots, I_{i-1}, I_i \cup I_{i+1}, I_{i+2}, \dots, I_r)$. Let $F' = \{\tau_1, \dots, \tau_{r-2}\}$ be its corresponding face in $\mathrm{Bary}(\Delta^{n-2})$. Note that $\tau_j = \sigma_j$ when $j < i$, and $\tau_j = \sigma_{j+1}$ when $j \geq i$. Thus, $F' \subseteq F$ and $|F'| = |F| - 1$. We conclude that F covers F' in $\mathrm{Bary}(\Delta^{n-2})$. $\square$

Lemma 3.3.6. *The Scarf complex Δ_M of the initial ideal M is* $\mathrm{Bary}(\Delta^{n-2})$. *Hence, we have $\Delta_M \cong$* Cyc.

Proof. We can identify generators of M with the set $\{\sigma \in \mathbb{N}^n \mid \sigma_n = 0, \vec{\mathbf{0}} \lneq \sigma \leq \vec{\mathbf{1}}\}$, which is the same as $\{\sigma \in \mathbb{N}^{n-1} \mid \vec{\mathbf{0}} \lneq \sigma \leq \vec{\mathbf{1}}\}$. We will use the two sets interchangeably to represent the generators of M. In this way, Δ_M and $\mathrm{Bary}(\Delta^{n-2})$ share the same set of vertices. We show that the two simplicial complexes are isomorphic by showing a bijection of facets.

Let $F = \{\sigma_1, \dots, \sigma_{n-1}\} \in \mathrm{Bary}(\Delta^{n-2})$ be a facet with $|\sigma_i| = i$. Let τ_i be the index of the only nonzero entry of $\sigma_i - \sigma_{i-1}$, where we set $\sigma_0 = \vec{\mathbf{0}}$. Using the isomorphism $\mathrm{Bary}(\Delta^{n-2}) \cong \mathrm{Cyc}$ in Lemma 3.3.5, we see that

$$\begin{aligned} s_F &= \mathrm{lcm}(\{x^{(\widetilde{C}\sigma_i)^+} \mid 1 \leq i \leq n-1\}) \\ &= x^{\tau_1 \to [n] \backslash \{\tau_1\}} x^{\tau_2 \to [n] \backslash \{\tau_1, \tau_2\}} \cdots x^{\tau_{n-2} \to \{\tau_{n-1}, n\}} x^{\tau_{n-1} \to \{n\}} \\ &= x_{\tau_1}^{-\sum_{j \neq \tau_1} \widetilde{C}_{\tau_1, j}} x_{\tau_2}^{-\sum_{j \neq \tau_1, \tau_2} \widetilde{C}_{\tau_2, j}} \cdots x_{\tau_{n-2}}^{-\sum_{j \in \{\tau_{n-1}, n\}} \widetilde{C}_{\tau_{n-2}, j}} x_{\tau_{n-1}}^{-\widetilde{C}_{\tau_{n-1}, n}}. \end{aligned}$$

If there is another face F' with $s'_F = s_F$, we can easily conclude that $F = F'$. So s_F is unique, and F is a face in Δ_M. By Lemma 2.3.9, F is a facet.

Now let F be a facet in Δ_M. We need to show three things: 1) If there are $\sigma, \tau \in F$ such that $|\sigma| = |\tau|$, then $\sigma = \tau$; 2) If there are $\sigma, \tau \in F$ such that $|\sigma| = |\tau| + 1$, then $\sigma - \tau = e_i$ for some i; 3) $|F| = n - 1$. It then follows that F is a facet in $\mathrm{Bary}(\Delta^{n-2})$.

If we have $\sigma, \tau \in F$ such that $|\sigma| = |\tau|$ but $\sigma \neq \tau$, we want to show that $\{\sigma, \tau\}$ is not a face, which contradicts the assumption that F is a face. Indeed, consider $\xi = \max(\sigma, \tau)$. Note that

$$s_{\{\sigma,\tau\}} = \mathrm{lcm}(x^{(\widetilde{C}\sigma)^+}, x^{(\widetilde{C}\tau)^+}) = \mathrm{lcm}\left(\prod_{\sigma_i \neq 0} x_i^{-\sum_{\sigma_j = 0} \widetilde{C}_{i,j}}, \prod_{\tau_i \neq 0} x_i^{-\sum_{\tau_j = 0} \widetilde{C}_{i,j}} \right),$$

and also

$$x^{(\widetilde{C}\xi)^+} = \prod_{\xi_i \neq 0} x_i^{-\sum_{\xi_j = 0} \widetilde{C}_{i,j}}.$$

So each exponent of x_i in $x^{(C\xi)^+}$ is always no bigger than the corresponding exponent in $s_{\{\sigma,\tau\}}$, and $s_{\{\sigma,\tau\}} = s_{\{\sigma,\tau,\xi\}}$. Therefore, $\{\sigma,\tau\}$ is not a face in Δ_M.

Suppose that $|\sigma| = |\tau| + 1$ but $\sigma - \tau$ is not a basis vector. Let $\xi = \max(\sigma,\tau)$. Similarly, $s_{\{\sigma,\tau\}} = s_{\{\sigma,\tau,\xi\}}$, and $\{\sigma,\tau\}$ cannot be a face of Δ_M.

Recall from Lemma 2.3.9 that $|F| \leq n-1$. We will prove by contradiction that $|F| = n-1$. Using the previous part, we can write $F = \{\sigma_1,\ldots,\sigma_l\}$ such that $|\sigma_1| < |\sigma_2| < \cdots < |\sigma_l|$. Suppose instead that $l < n-1$. Then there is some k such that $(\sigma_i)_k = 0$ for all i. Put $\xi = \sigma_l + e_k$. We show that $F \cup \{\xi\}$ is a face of Δ_M, which contradicts the assumption that F is a facet. One can check that

$$s_{F\cup\{\xi\}} = s_F \cdot x_k^{-\sum_{\xi_j=0} \widetilde{C}_{k,j}},$$

which is unique amongst all faces of Δ_M. $\square$

Let $\mathcal{CYC}$ denote the following complex of free R-modules:

$$0 \longleftarrow R^{\mathrm{Cyc}_{n,1}} \longleftarrow R^{\mathrm{Cyc}_{n,2}} \longleftarrow \cdots \longleftarrow R^{\mathrm{Cyc}_{n,n}} \longleftarrow 0,$$

where the boundary map from $R^{\mathrm{Cyc}_{n,r}}$ to $R^{\mathrm{Cyc}_{n,r-1}}$ is given by

$$(I_1,\ldots,I_r) \mapsto \begin{array}{l} \sum_{s=1}^{r-1}(-1)^{s-1}x^{I_s\to I_{s+1}}(I_1,\ldots,I_{s-1},I_s\cup I_{s+1},I_{s+2},\ldots,I_r) \\ -x^{I_r\to I_1}(I_2,I_3,\ldots,I_{r-1},I_1\cup I_r). \end{array}$$

We show in the following theorem that this is the minimal free resolution of R/I.

Theorem 3.3.7. *The complex $\mathcal{CYC}$ coincides with the algebraic Scarf complex of I, and it minimally resolves R/I.*

Remark 3.3.8. In [21], the authors showed that the above complex gives the minimal free resolution of the lattice ideal corresponding to the full Laplacian of an undirected and saturated multigraph. Here we explicitly show that the same thing holds for directed graphs that are saturated. In terms of M-matrices, given an M-matrix A and its (u,w)-extension $\widetilde{A}$, this theorem applies to the lattice ideal $I(\widetilde{A})$ if the following two conditions are satisfied: 1. $u = w = \vec{\mathbf{1}}$; 2. The graph associated to $\widetilde{A}$ is saturated. These two conditions together ensure that the binomials $\{x^{I\to[n]\setminus I} - x^{[n]\setminus I\to I} \mid I \subseteq [n-1]\}$ is a minimal Gröbner basis for $I(\widetilde{A})$.

Proof. Since I is generic, by Theorem 2.3.23 and Corollary 2.3.24, we only need to find the Scarf complex of $M = \mathrm{In}(I)$ and apply the combinatorial rule to know the minimal free resolution of R/I. Lemma 3.3.6 says that $\Delta_M \cong \mathrm{Bary}(\Delta^{n-2}) \cong \mathrm{Cyc}$.

Consider the partition $(I_1,\ldots,I_r) \in \mathrm{Cyc}_{n,r}$. Let $F = \{\sigma_1,\ldots,\sigma_r\}$ denote the corresponding face in Δ_M. For any $1 \leq s < r$, the face $F\setminus\{\sigma_s\}$ corresponds to the partition $(I_1,\ldots,I_{s-1},I_s\cup I_{s+1},I_{s+2},\ldots I_r)$. One can check that

$$\frac{s_F}{s_{F\setminus\{\sigma_s\}}} = x^{I_s\to I_{s+1}},$$

and that $\operatorname{sign}(s,F)=(-1)^{s-1}$. Therefore, the boundary map without the last term gives the algebraic Scarf complex of M.

Now we use the combinatorial rule described in Theorem 2.3.23 to show that $(I_2,\dots,I_{r-1},I_r\cup I_1)$ is the additional face covered once we move to the Scarf complex of I. The face F in $\operatorname{Bary}(\Delta^{n-2})$ gives us a subset of generators of M, and hence generators of I. For $1\le l<r$, the corresponding generator of I is

$$x^{(\widetilde{C}\sigma_l)^+}-x^{(\widetilde{C}\sigma_l)^-}=x^{\bigcup_{i=1}^{l} I_i\to\bigcup_{i=l+1}^{r} I_i}-x^{\bigcup_{i=l+1}^{r} I_i\to\bigcup_{i=1}^{l} I_i}.$$

Recall the least common multiple s_F of the leading terms:

$$s_F=\operatorname{lcm}(\{x^{(\widetilde{C}\sigma)^+}\mid \sigma\in F\})=x^{I_1\to[n]\setminus I_1}x^{I_2\to([n]\setminus(I_1\cup I_2))}\cdots x^{I_{r-1}\to I_r}.$$

Now we want to compute, for $1\le l<r$, the monomials s_l:

$$s_l=\frac{s_F}{x^{(\widetilde{C}\sigma_l)^+}}x^{(\widetilde{C}\sigma_l)^-}=\frac{s_F}{x^{\bigcup_{i=1}^{l} I_i\to\bigcup_{i=l+1}^{r} I_i}}\cdot x^{\bigcup_{i=l+1}^{r} I_i\to\bigcup_{i=1}^{l} I_i}$$

To see some examples:

$$\begin{aligned}
s_1&=\frac{s_F}{x^{I_1\to[n]\setminus I_1}}x^{[n]\setminus I_1\to I_1}\\
&=x^{[n]\setminus I_1\to I_1}x^{I_2\to[n]\setminus(I_1\cup I_2)}\cdots x^{I_{r-1}\to I_r}\\
&=x^{I_r\to I_1}x^{I_2\to I_1}\cdots x^{I_{r-1}\to I_1}x^{I_2\to[n]\setminus(I_1\cup I_2)}\cdots x^{I_{r-1}\to I_r}\\
&=x^{I_r\to I_1}x^{I_2\to[n]\setminus I_2}x^{I_3\to[n]\setminus(I_2\cup I_3)}\cdots x^{I_{r-1}\to(I_r\cup I_1)}.
\end{aligned}$$

$$\begin{aligned}
s_2&=\frac{s_F}{x^{(I_1\cup I_2)\to[n]\setminus(I_1\cup I_2)}}x^{[n]\setminus(I_1\cup I_2)\to(I_1\cup I_2)}\\
&=x^{[n]\setminus(I_1\cup I_2\to(I_1\cup I_2)}x^{I_1\to I_2}x^{I_3\to[n]\setminus\bigcup_{i=1}^{3} I_i}\cdots x^{I_{r-1}\to I_r}\\
&=x^{I_r\to(I_1\cup I_2)}x^{I_3\to(I_1\cup I_2)}x^{I_4\to(I_1\cup I_2)}\cdots x^{I_{r-1}\to(I_1\cup I_2)}x^{I_1\to I_2}x^{I_3\to[n]\setminus\bigcup_{i=1}^{3} I_i}\cdots x^{I_{r-1}\to I_r}\\
&=x^{I_r\to(I_1\cup I_2)}x^{I_1\to I_2}x^{I_3\to[n]\setminus I_3}x^{I_4\to[n]\setminus(I_3\cup I_4)}\cdots x^{I_{r-1}\to(I_r\cup I_1\cup I_2)}.
\end{aligned}$$

We can see a general pattern that s_l is not divisible by x_i if $i\in I_l$. So

$$\gcd(\{s_l\mid 1\le l<r\})=\gcd(\{x^{I_r\to\bigcup_{i=1}^{l} I_i}\mid 1\le l<r\})=x^{I_r\to I_1}.$$

Let $F'\in\operatorname{Bary}(\Delta^{n-2})$ be the face corresponding to the partition $(I_2,\cdots,I_{r-1},I_r\cup I_1)$. Observe that

$$s_{F'}=x^{I_2\to[n]\setminus I_2}x^{I_3\to[n]\setminus I_1\cup I_2}\cdots x^{I_{r-1}\to I_r\cup I_1}=\frac{s_1}{x^{I_r\to I_1}}.$$

Therefore, the additional face covered by $(I_1,\dots,I_r)$ is $(I_2,\dots,I_{r-1},I_r\cup I_1)$. Also, note that

$$\begin{aligned}
\gcd(\{x^{(\widetilde{C}\sigma_l)^-}\mid 1\le l<r\})&=\gcd(\{x^{\bigcup_{i=l+1}^{r} I_i\to\bigcup_{i=1}^{l} I_i}\mid 1\le l<r\})\\
&=\gcd(\{x^{I_r\to\bigcup_{i=1}^{l} I_i}\mid 1\le l<r\})\\
&=x^{I_r\to I_1}.
\end{aligned}$$

So the additional term in the boundary map is of the form described in the theorem.

Finally, we need to show that $\partial^2 = 0$. Let us start with some partition $(I_1, \ldots, I_r)$. By definition, we have

$$\begin{aligned}\partial(I_1, \ldots, I_r) = &\sum_{s=1}^{r-1} (-1)^{s-1} x^{I_s \to I_{s+1}} (I_1, \ldots, I_{s-1}, I_s \cup I_{s+1}, I_{s+2}, \ldots, I_r) \\ &- x^{I_r \to I_1} (I_2, \ldots, I_{r-1}, I_r \cup I_1).\end{aligned}$$

Since the boundary map without the additional term gives the algebraic Scarf complex of Δ_M, to show that $\partial^2 = 0$, we only need to consider the behavior of the additional term. Thus, we have

$$\begin{aligned}\partial^2(I_1, \ldots, I_r) = &\, x^{I_1 \to I_2}(-1) x^{I_r \to I_1 \cup I_2} (I_3, I_4, \ldots, I_{r-1}, I_r \cup I_1 \cup I_2) \\ &+ \sum_{s=2}^{r-2} (-1)^{s-1} x^{I_s \to I_{s+1}} (-1) x^{I_r \to I_1} (I_2, \ldots, I_{s-1}, I_s \cup I_{s+1}, I_{s+2}, \ldots, I_r \cup I_1) \\ &+ (-1)^{r-2} x^{I_{r-1} \to I_r} (-1) x^{I_{r-1} \cup I_r \to I_1} (I_2, \ldots, I_{r-2}, I_{r-1} \cup I_r \cup I_1) \\ &- x^{I_r \to I_1} \left(\sum_{t=1}^{r-3} (-1)^{t-1} x^{I_{t+1} \to I_{t+2}} (I_2, \ldots, I_t, I_{t+1} \cup I_{t+2}, I_{t+3}, \ldots, I_r \cup I_1) \right) \\ &- x^{I_r \to I_1} (-1)^{r-2} x^{I_{r-1} \to I_r \cup I_1} (I_2, \ldots, I_{r-2}, I_{r-1} \cup I_r \cup I_1) \\ &+ x^{I_r \to I_1} x^{I_r \cup I_1 \to I_2} (I_3, I_4, \ldots, I_{r-1}, I_r \cup I_1 \cup I_2).\end{aligned}$$

One can then check that the coefficient for each partition is 0. $\square$

We end this section by giving an example when the graph is not saturated. In this case, the lattice ideal is not generic, but we can still obtain a minimal free resolution from the chain complex $\mathcal{CYC}$ introduced in Theorem 3.3.7.

Example 3.3.9. Let us return to C_3 and, without loss of generality, suppose that $\rho = m\rho_1$. Its extended McKay-Cartan matrix and the McKay-Cartan matrix are shown below:

$$\widetilde{C} = \begin{pmatrix} m & 0 & -m \\ -m & m & 0 \\ 0 & -m & m \end{pmatrix}, \qquad C = \begin{pmatrix} m & 0 \\ -m & m \end{pmatrix}.$$

Since now there is no edge going from v_1 to v_3, we get the ordering $x_1 > x_2$ and the monomial ordering on R is still the normal `grevlex`. Applying Theorem 3.3.7, the lattice ideal $I = I(\widetilde{C})$ is generated by the three binomials:

$$I = \langle x_1^m - x_2^m, x_2^m - x_3^m, x_1^m - x_3^m \rangle$$

But this is not a minimal generating set since two of the generators share the same leading term. This is expected as I is not generic. Note that the chain complex given by the theorem is still a free resolution, but it is no longer minimal as will be explained.

We record the free resolution as

$$0 \longleftarrow R \xleftarrow{\partial_1} R^3 \xleftarrow{\partial_2} R^2 \longleftarrow 0,$$

with the boundary maps being

$$\partial_1 = \begin{array}{cc} & \begin{array}{ccc} (1,23) & (2,13) & (12,3) \end{array} \\ (123) & \begin{bmatrix} x_1^m - x_2^m & x_2^m - x_3^m & x_1^m - x_3^m \end{bmatrix} \end{array},$$

$$\partial_2 = \begin{array}{cc} & \begin{array}{cc} (1,2,3) & (2,1,3) \end{array} \\ \begin{array}{c} (1,23) \\ (2,13) \\ (12,3) \end{array} & \begin{bmatrix} -1 & -x_3^m \\ -1 & -x_1^m \\ 1 & x_2^m \end{bmatrix} \end{array}.$$

Note that the first column of ∂_2 consists only of constants. By performing row and column operations on ∂_2, we can replace its first column and first row with basis vectors. Consider matrices U and V, where U encodes the row operations performed on ∂_2, and V the column operations:

$$U = \begin{bmatrix} -1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix}, \quad V = \begin{bmatrix} 1 & -x_3^m \\ 0 & 1 \end{bmatrix}.$$

The inverses of U and V are:

$$U^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}, \quad V^{-1} = \begin{bmatrix} 1 & x_3^m \\ 0 & 1 \end{bmatrix}.$$

Define $d_1 = \partial_1 U^{-1}$ and $d_2 = U \partial_2 V$. Their corresponding matrices are:

$$d_1 = \partial_1 U^{-1} = \begin{array}{cc} & \begin{array}{ccc} (1,23) & (2,13) & (12,3) \end{array} \\ (123) & \begin{bmatrix} 0 & x_2^m - x_3^m & x_3^m - x_1^m \end{bmatrix} \end{array},$$

$$d_2 = U \partial_2 V = \begin{array}{cc} & \begin{array}{cc} (1,2,3) & (2,1,3) \end{array} \\ \begin{array}{c} (1,23) \\ (2,13) \\ (12,3) \end{array} & \begin{bmatrix} 1 & 0 \\ 0 & x_3^m - x_1^m \\ 0 & x_3^m - x_2^m \end{bmatrix} \end{array}.$$

Therefore, we have a commutative diagram as shown below. Since the vertical maps are all isomorphisms, the top row is also exact and hence a free resolution of I.

$$\begin{tikzcd} 0 & R \arrow[l] & R^3 \arrow[l, "d_1"'] & R^2 \arrow[l, "d_2"'] & 0 \arrow[l] \\ 0 & R \arrow[l] \arrow[u, "="] & R^3 \arrow[l, "\partial_1"] \arrow[u, "U"] & R^2 \arrow[l, "\partial_2"] \arrow[u, "V^{-1}"] & 0 \arrow[l] \end{tikzcd}$$

This resolution is not minimal since the partition $(1,2,3)$ is mapped directly to $(1,23)$, which is in the kernel of d_1. We can therefore remove the generator $(1,23)$ from R^3 and $(1,2,3)$ from R^2 to reduce the ranks by 1 and obtain the following exact sequence:

$$0 \longleftarrow R/I \longleftarrow R \xleftarrow{d_1'} R^2 \xleftarrow{d_2'} R \longleftarrow 0,$$

where d_1', d_2' are matrices with the corresponding row and column removed. Namely, they are

$$d_1' = \begin{array}{c c} & \begin{array}{cc} (2,13) & (12,3) \end{array} \\ (123) & \begin{bmatrix} x_2^m - x_3^m & x_3^m - x_1^m \end{bmatrix} \end{array}, \quad d_2' = \begin{array}{c c} & (2,1,3) \\ \begin{array}{c} (2,13) \\ (12,3) \end{array} & \begin{bmatrix} x_3^m - x_1^m \\ x_3^m - x_2^m \end{bmatrix} \end{array}.$$

This resolution is similar to a cellular resolution supported on the 1-simplex as shown in Figure 3.3, with vertices labeled by the cyclic partitions. Also, note that

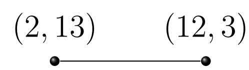

Figure 3.3: The 1-simplex.

this complex can also be obtained from Figure 3.1 by removing the leftmost vertex and the corresponding edge. How this phenomena generalizes to larger n would be an interesting problem for future investigation.

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