Kasteleyn's Tiling Theorem Using Complex Weights

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Abstract

In 1961, Fisher, and Temperley [11], and Kasteleyn [4], independently provided a formula for the number of possible tilings of a $2m \times n$ checkerboard with 1×2 dominos. An essential step in typical proofs of this formula involves computing the eigenvalues of a matrix associated with the checkerboard. This thesis extends Chen's undergraduate thesis [3], which investigated combinatorial strategies for a potential proof of Kasteleyn's formula involving complex-weighted *pseudo-tilings*. A main result of this thesis is to find a relation between these weights and Chebyshev polynomials of the third kind.

Introduction

A tiling of a region is a collection of tiles that covers the entire region without any overlap. There are many questions to ask: for example, given a region and a set of tiles, does a tiling of the region exist? If so, what is the number of different tilings of the region? Such problems are broadly referred to as tiling problems. A good overview of the subject is provided by [1].

This thesis is interested in domino tilings of a checkerboard. See Figure 1 for an example. This topic has long been a research interest for physicists, since such checkerboard tilings, also known as the dimer models, are solvable models that exhibited certain types of phase transitions [5]. A well-known result is the number of ways to tile a $m \times n$ checkerboard with 2×1 and 1×2 dominoes, discovered independently by Kasteleyn [4] and by Fisher and Temperley [11] in 1961:

$$\prod_{v=1}^{m}\prod_{h=1}^{n}\left(4\cos^2\left(\frac{v\pi}{m+1}\right)+4\cos^2\left(\frac{h\pi}{n+1}\right)\right)^{\frac{1}{4}}.$$

Kasteleyn later extended his enumeration of domino tilings to any bipartite planar graph. A general reference including an extensive history is provided by [8].

Kasteleyn's formula is the motivation of this thesis. Given Kasteleyn's formula, a standard idea (presented in Chapter 1) converts the problem into the calculation

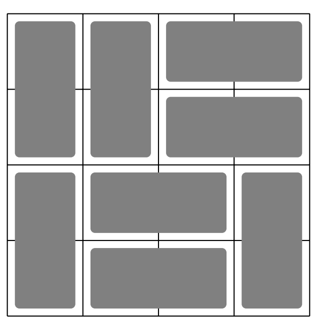


Figure 1: A domino tiling of a 4×4 checkerboard.

of the determinant of a matrix [9]. From this point of view, Kasteleyn's formula is the product of the eigenvalues of that matrix, thus the proof of it is mostly algebraic. The undergraduate thesis of Chen [3] outlines a path to a possible alternative proof of Kasteleyn's formula in the $2m \times 2n$ case.

Chen introduces the idea of *pseudo-dominoes*, which can be white or black, facing any of the four directions: left, right, up or down. A *pseudo-tiling* is a collection of pseudo-dominoes in every other square on the checkerboard, see Figure 2.1. Note that in a pseudo-tiling, pseudo-dominoes can hang over the edge of the board and overlap with each other. In Chapter 2, we talk about a subset of these pseudo-tilings called *proper pseudo-tilings*, which has a bijection to the standard tilings.

Chen assigns weights to these pseudo-dominoes. The weight of a pseudo-tiling is the product of all weights of pseudo-dominoes in it. The weight of a set of pseudo-tilings is the sum of all weights of pseudo-tilings in it. By construction, the weight of the set of all possible pseudo-tilings is the number of proper tilings, which is given by Kasteleyn's formula. Chen and Benjamin aim to give an independent combinatorial proof that the weight of *improper pseudo-tilings* is 0, which would than provide a new combinatorial proof of Kasteleyn's formula. Chen succeeds in proving this result for a special subset of $2 \times 2n$ pseudo-tilings (see Proposition 2.2.8). This thesis extends Chen's result, see Chapter 4.

In Chapter 1, we revisit the original Kasteleyn's Formula and give a standard proof of the original theorem. It is also known that the number of $2 \times n$ tilings is the Fibonacci number. Chapter 1 also gives an algebraic proof that the Fibonacci number is equivalent to Kasteleyn's Formula in the $2 \times n$ case. Though similar proofs have been given before by Webb and Parberry [14] and by Sury [10], this thesis gives a proof more in the flavor of Kasteleyn's formula.

We start Chapter 2 by reversing what we do at the end of Chapter 1. Using the recursion of *Fibonacci polynomials*, we give a combinatorial proof of Kasteleyn's formula in the $2 \times n$ case. Then, we describe Chen and Benjamin's idea for a possible combinatorial proof of Kasteleyn's formula using pseudo-tilings with complex-weighted pseudo-dominoes. Proposition 2.2.8 presents Chen's result that the weight of a subset of $2 \times 2n$ improper pseudo-tilings is 0.

Removing those pseudo-tilings covered by Chen's result leaves us a subset of $2 \times 2n$ pseudo-tilings. We write this set as a disjoint union of $X_n \bigsqcup Y_n$, where X_n consists of improper pseudo-tilings in the set and Y_n consists of proper ones. Each element of Y_n has weight 1, and the elements of Y_n are in bijection with the complete set of standard domino tilings of the $2 \times 2n$ checkerboard. Thus, to complete Chen and Benjamin's program for the $2 \times 2n$ case, we must give a combinatorial proof that the weight of X_n is 0. To that end, in Chapter 3, we form a partition $X_n = \bigsqcup_h X_{n,h}$ where $X_{n,h}$ is the set of elements in X_n having exactly h horizontal pseudo-dominoes.

A main result of the thesis is Theorem 3.1.1, which shows that the weight of $X_{n,h}$

is 0. The proof is mainly algebraic, using *Chebyshev polynomials of the third kind*, $V_n(x)$. The connection between Chebyshev polynomials and Fibonacci numbers has been well studied in [2]. The proof of Theorem 3.1.1 depends on the novel connection between $V_n(x)$ and the complex weights of pseudo-tilings. We find that $V_n(x)$ and their derivatives determine the generating function of the weights of pseudo-tilings according to the number of white *horizontal* pseudo-dominoes. In Proposition 3.6.1, by Taylor series expansion, we show that $V_n(\frac{x}{2}+1)$ is the generating function of the number of proper pseudo-tilings according to the number of *vertical* pseudo-dominoes.

In Section 3.4, we prove the number of standard $2 \times 2n$ tilings with exactly $2h$ horizontal dominoes is the binomial coefficient $\binom{2n-h}{h}$. We do so in two ways, one using a stars and bars method and another using a new bijection between Dyck paths of a certain type and pseudo-tilings.

At the end of Chapter 3, we give an algebraic proof that the weight of $X_{n,h}$ is 0. Our ultimate goal, however, is to give an independent, more combinatorial proof of this fact, in the spirit of the proof given by Chen in Proposition 2.2.8. In Chapter 4, we succeed in the task for $X_{n,1}$, $X_{n,2}$. We could not see how to extend those ideas to the case of $X_{n,h}$ for $h > 2$. However, we do present a new idea motivated by Nurul [13] that handles the case $X_{n,n}$.This idea might extend to give new proof of Kasteleyn's formula in the general $2m \times n$ case.

Chapter 1

Kasteleyn's Formula

1.1 Statement of Kasteleyn's Theorem

In this chapter, we present a proof of Kasteleyn's formula [4] for the number of tilings in a checkerboard. Our proof is a variation of the ideas posted by Noam Elkies to the rec.puzzles Google group.

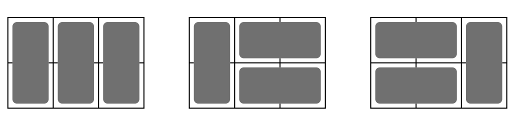
Let C be an $m \times n$ checkerboard. If C can be tiled by 2×1 and 1×2 dominos, then mn is an even number. Without loss of generality, we assume m is even. Define $d = mn/2$. For integers h, v, define $t(h, v)$ to be the number of tilings of C with h horizontal tiles and v vertical tiles, and consider the *tiling generating function*

$$T_C(x, y) = \sum_{h,v} t(h, v) x^v y^h.$$

Theorem 1.1.1 (Kasteleyn, 1961)**.** *Let C be an $m \times n$ checkerboard with m an even number. Then the tiling generating function $T_C(x, y)$ for C is,*

$$\begin{aligned} T_C(x,y) &= \prod_{v=1}^{m}\prod_{h=1}^{n}\left(4x^2\cos^2\left(\frac{v\pi}{m+1}\right)+4y^2\cos^2\left(\frac{h\pi}{n+1}\right)\right)^{\frac{1}{4}} \\ &= \begin{cases} \prod_{v=1}^{\frac{m}{2}}\prod_{h=1}^{\frac{n}{2}}\left(4x^2\cos^2\left(\frac{v\pi}{m+1}\right)+4y^2\cos^2\left(\frac{h\pi}{n+1}\right)\right), & \text{if } n \text{ is even,} \\ x^{\frac{m}{2}}\prod_{v=1}^{\frac{m}{2}}\prod_{h=1}^{\frac{n-1}{2}}\left(4x^2\cos^2\left(\frac{v\pi}{m+1}\right)+4y^2\cos^2\left(\frac{h\pi}{n+1}\right)\right), & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

Example 1.1.2. Let C be a 2×3 checkerboard. There are 3 ways to tile C:



x^3 $\quad$ xy^2 $\quad$ xy^2

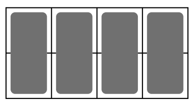
The tiling generating function $T_C(x, y)$ is

$$T_C(x,y) = \sum_{h,v} t(h,v) x^v y^h = x^3 + 2xy^2,$$

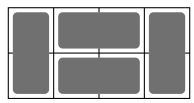
which agrees with the formula in Theorem 1.1.1, for the case when $n = 3$ is odd:

$$\begin{aligned}
T_C(x,y) &= x^{\frac{m}{2}} \prod_{v=1}^{\frac{m}{2}} \prod_{h=1}^{\frac{n-1}{2}} \left(4x^2 \cos^2\left(\frac{v\pi}{m+1}\right) + 4y^2 \cos^2\left(\frac{h\pi}{n+1}\right) \right) \\
&= x(x^2 + 4 \cdot \frac{1}{2} y^2) \\
&= x^3 + 2xy^2.
\end{aligned}$$

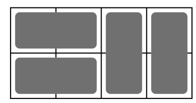
Example 1.1.3. Now let C be a 2×4 checkerboard. There are 5 ways to tile C:



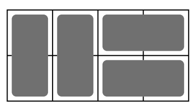
x^4



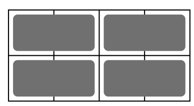
x^2y^2



x^2y^2



x^2y^2



y^4

.

This also agrees with Kasteleyn's formula. Note that here $n = 4$ is even.

$$\begin{aligned}
T_C(x,y) &= \left(4x^2 \cos^2 \frac{\pi}{3} + 4y^2 \cos^2 \frac{\pi}{5}\right)\left(4x^2 \cos^2 \frac{\pi}{3} + 4y^2 \cos^2 \frac{2\pi}{5}\right) \\
&= \left(x^2 + 4y^2 \left(\frac{\omega_5 + \omega_5^4}{2}\right)^2\right)\left(x^2 + 4y^2\left(\frac{\omega_5^2 + \omega_5^3}{2}\right)^2\right) \\
&= x^4 + [(\omega_5 + \omega_5^4)^2 + (\omega_5^2 + \omega_5^3)^2]x^2y^2 + (\omega_5 + \omega_5^4)^2(\omega_5^2 + \omega_5^3)^2 y^4 \\
&= x^4 + (\omega_5^2 + 2 + \omega_5^3 + \omega_5^4 + 2 + \omega_5)x^2y^2 + (\omega_5^2 + 2 + \omega_5^3)(\omega_5^4 + 2 + \omega_5)y^4 \\
&= x^4 + 3x^2y^2 + y^4,
\end{aligned}$$

where ω_5 is the fifth root of unity. In the second to last line, we used the fact that

$$\omega_5 + \omega_5^2 + \omega_5^3 + \omega_5^4 = -1.$$

Theorem 1.1.1 can give the number of tilings of C directly.

Corollary 1.1.4. *1. If C is an $m \times n$ checkerboard with the parity of m and n arbitrary, then the number of tilings of C is*

$$\prod_{j=1}^{m}\prod_{k=1}^{n}\left(4\cos^2\frac{j\pi}{m+1}+4\cos^2\frac{k\pi}{n+1}\right)^{\frac{1}{4}}. \tag{1.1}$$

2. If C is a $2m \times 2n$ checkerboard, then the number of tilings of C is

$$\prod_{j=1}^{m}\prod_{k=1}^{n}\left(4\cos^2\frac{j\pi}{2m+1}+4\cos^2\frac{k\pi}{2n+1}\right). \tag{1.2}$$

Proof. 1. Let $x = y = 1$ in Theorem 1.1.1. Then this formula gives the number of tilings T_C of any arbitrary $m \times n$ checkerboard C.

$$T_C = \prod_{j=1}^{m}\prod_{k=1}^{n}\left(4\cos^2\frac{j\pi}{m+1}+4\cos^2\frac{k\pi}{n+1}\right)^{\frac{1}{4}}.$$

Note that for mn an odd number, $T_C = 0$. This is also true in the above formula. For the term where $j = \frac{m+1}{2}, k = \frac{n+1}{2}$, we have

$$4\cos^2\frac{j\pi}{m+1}+4\cos^2\frac{k\pi}{n+1} = 4\cos^2\frac{\pi}{2}+4\cos^2\frac{\pi}{2} = 0,$$

which makes the whole product 0.

2. Now, let C be a $2m \times 2n$ checkerboard. Then using the results from the previous part, T_C is

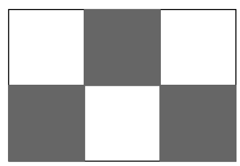
$$\begin{aligned} T_C &= \prod_{j=1}^{2m}\prod_{k=1}^{2n}\left(4\cos^2\frac{j\pi}{m+1}+4\cos^2\frac{k\pi}{n+1}\right)^{\frac{1}{4}} \\ &= \prod_{j=1}^{m}\prod_{k=1}^{n}\left(4\cos^2\frac{j\pi}{m+1}+4\cos^2\frac{k\pi}{n+1}\right). \end{aligned}$$

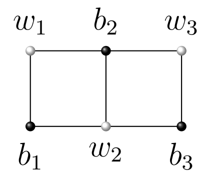
□

Overview of Proof of Theorem 1.1.1. In the rest of this chapter, our goal is to prove Theorem 1.1.1. We need to first find a $d \times d$ matrix A whose permanent (defined below) is $T_C(x, y)$, where $d = \frac{mn}{2}$. However, permanents are hard to calculate, so we modify A to get a weighted matrix A' whose determinant is $\mathrm{perm}(A)$ (up to a manageable constant). Next, we want to compute $\det(A')$. Doing it directly is still difficult, so we relate A' to the weighted adjacency matrix K of an $m \times n$ grid graph. Finally, we calculate $\det(K)$ by giving the eigenvalues of K.

1.2 Preliminaries

Let C be an $m \times n$ checkerboard. We make every square in the checkerboard a vertex of a planar graph and the vertices inherit the colors of the squares. Two vertices are connected if and only if their corresponding squares share an edge. This transform gives a planar graph G_C with d black vertices and d white vertices. Label the white vertices $w_1, \cdots, w_d$ and the black vertices $b_1, \cdots, b_d$. The ordering is arbitrary, although the left-right and top-down ordering is standard. An example of a 2×3 checkerboard is given below:

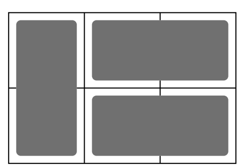


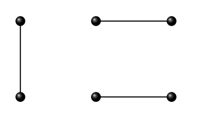


2×3 checkerboard C Planar graph G_C

Definition 1.2.1. A *perfect matching* of a graph is a matching in which every vertex of the graph is incident to exactly one edge of the matching.

There is a one-to-one correspondence between a perfect matching of G_C and a tiling of C. The following example of the previous C and G_C shows the obvious bijection:





A tiling of C A perfect matching of G_C

Definition 1.2.2. For $m \geq 2$, a *path graph* P_m is the connected graph with exactly 2 vertices of degree 1 and $m - 2$ vertices of degree 2.

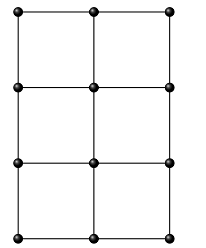
Definition 1.2.3. If P_m and P_n are two path graphs with vertex sets U (m elements) and V (n elements), and edge sets E_{P_m} and E_{P_n} respectively, then we define the *Cartesian product* $P_m \square P_n$ to be the graph with,

- Vertex set $U \times V$;
- Two vertices (u_1, v_1) and (u_2, v_1) are connected by an edge if and only if $(u_1, u_2) \in E_{P_m}$. Similarly, (u_1, v_1) and (u_1, v_2) are connected by an edge if and only if $(v_1, v_2) \in E_{P_n}$.

Example 1.2.4. The graph of the Cartesian product of $P_4 \square P_3$ is shown below:

P_4

P_3



$P_4 \square P_3$

Notice that the graph $P_m \square P_n$ is exactly the planar graph G_C.

Let $B(P_m)$ and $B(P_n)$ be the adjacency matrices of P_m and P_n.

Lemma 1.2.5. *For each* $k = 1, \cdots, n$,

$$v_k = (v_{k,1}, v_{k,2}, \cdots, v_{k,n})^t = \left(\sin \frac{k\pi}{n+1}, \sin \frac{2k\pi}{n+1}, \cdots, \sin \frac{2nk\pi}{n+1} \right)^t \in \mathbb{R}^n$$

is an eigenvector for $B(P_n)$, *with corresponding eigenvalue*

$$\mu_k = 2 \cos \frac{k\pi}{n+1}.$$

Proof. The adjacency matrix $B(P_n)$ has the form

$$B(P_n) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 1 & 0 & \cdots & 0 \\ & & & \cdots & & & \\ 0 & & \cdots & 0 & 1 & 0 & 1 \\ 0 & & \cdots & & 0 & 1 & 0 \end{pmatrix}.$$

We have,

$$\begin{aligned} v_{k,j-1} + v_{k,j+1} &= \sin \frac{(j-1)k\pi}{n+1} + \sin \frac{(j+1)k\pi}{n+1} \\ &= \sin \left(\frac{jk\pi}{n+1} - \frac{k\pi}{n+1} \right) + \sin \left(\frac{jk\pi}{n+1} + \frac{k\pi}{n+1} \right) \\ &= 2 \cos \frac{k\pi}{n+1} \sin \frac{jk\pi}{n+1} \\ &= \mu_k v_{k,j}. \end{aligned}$$

Thus, $B(P_n) v_k = \mu_k v_k$, for $k = 1, \cdots, n$. $\square$

Remark 1.2.6. One may discover the eigenvector and eigenvalues in the above lemma by solving a system of recurrences. Suppose $u = (u_1, \cdots, u_n)^t$ is an eigenvector with engenvalue μ. Then

$$\begin{cases} u_2 = \mu u_1 \\ u_1 + u_3 = \mu u_2 \\ u_2 + u_4 = \mu u_3 \\ \cdots \\ u_{n-1} = \mu u_n \end{cases} . \tag{1.3}$$

Equation (1.3) is a linear recurrence of

$$\begin{cases} u_0 := 0, \\ u_{j+1} + u_{j-1} = \mu u_j, \quad \text{for } 1 \le j \le n, \\ u_{n+1} := 0. \end{cases} \tag{1.4}$$

The above linear recurrence gives the following characteristic equation

$$x^2 - \mu x + 1 = 0. \tag{1.5}$$

Solving the above linear recurrence and the characteristic equation, we would obtain the eigenvectors and their corresponding eigenvalues as stated in the lemma.

Definition 1.2.7. Let x and y be indeterminates and define a $d \times d$ *matrix* $A = A_C$ *for the planar graph* G_C by

$$A_{ij} = \begin{cases} x, & \text{if } (w_i, b_j) \text{ is a vertical edge of } G_C, \\ y, & \text{if } (w_i, b_j) \text{ is a horizontal edge of } G_C, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 1.2.8. The *permanent* of a $k \times k$ matrix M with entries $(a_{i,j})$ is

$$\mathrm{perm}(M) = \sum_{\sigma \in S_k} \prod_{i=1}^{k} a_{i,\sigma(i)},$$

where $\sigma(i)$ is an element of the symmetric group S_k, i.e, all permutations of the set $\{1, 2, \cdots, k\}$.

Example 1.2.9. For the 2×3 checkerboard in the previous example, the matrix A of its planar graph G_C is

$$A = \begin{pmatrix} x & y & 0 \\ y & x & y \\ 0 & y & x \end{pmatrix}.$$

The permanent of the matrix A is

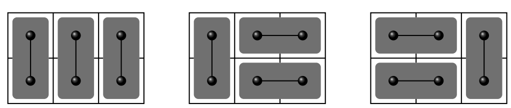
$$\begin{aligned}\operatorname{perm}(A) &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ &\quad + a_{13}a_{22}a_{31} + a_{12}a_{21}a_{33} + a_{11}a_{23}a_{32} \\ &= x^3 + 0 + 0 + 0 + xy^2 + xy^2 \\ &= x^3 + 2xy^2.\end{aligned}$$

It follows that the permanent of A is the tiling generating function for C.

Proposition 1.2.10.

$$\operatorname{perm}(A) = \sum_{\sigma \in S_d} \prod_{i=1}^{d} \operatorname{wt}(w_i, b_{\sigma(i)}) = T_C(x, y). \tag{1.6}$$

Proof. The σ-th term in $\operatorname{perm}(A)$ is nonzero if and only if the edges $(w_i, b_{\sigma(i)})$ form a perfect matching for G_C. It is easy to see that there is a one-to-one correspondence between perfect matchings of G_C and tilings of C. The example of a 2×3 checkerboard is given below,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x^3 xy^2 xy^2

□

Definition 1.2.11. With notation as above, the *weight* of each vertex pair (w_i, b_j) is defined by

$$\operatorname{wt}(w_i, b_j) = \begin{cases} ix, & \text{if } (w_i, b_j) \text{ is a vertical edge,} \\ y, & \text{if } (w_i, b_j) \text{ is a horizontal edge,} \\ 0, & \text{otherwise.} \end{cases}$$

The *weighted $d \times d$ matrix* $A' = A'_C$ is defined as $A'_{ij} = \operatorname{wt}(w_i, b_j)$.

Example 1.2.12. The weighted matrix A' in the previous example is

$$A' = \begin{pmatrix} ix & y & 0 \\ y & ix & y \\ 0 & y & ix \end{pmatrix}.$$

Let $\{f_i\}$ and $\{f'_i\}$ be bases for $\mathbb{R}^n$ and $\mathbb{R}^{n'}$ respectively, and let $\{e_i\}$ and $\{e'_i\}$ be the bases for $\mathbb{R}^m$ and $\mathbb{R}^{m'}$ respectively. If $s \in \mathbb{R}^m$ and $t \in \mathbb{R}^n$ are vectors, then the *tensor product* $s \otimes t = \sum_{i,j} s_i t_j (e_i \otimes f_j)$. If $S\colon \mathbb{R}^m \to \mathbb{R}^{m'}$ and $T\colon \mathbb{R}^n \to \mathbb{R}^{n'}$ are matrices. Then the *tensor product* $S \otimes T : \mathbb{R}^m \otimes \mathbb{R}^n \to \mathbb{R}^{m'} \otimes \mathbb{R}^{n'}$ is a matrix $(S \otimes T)(e_i \otimes f_j) = S(e_i) \otimes T(f_j)$. We choose the *lexicographic ordering* for the bases $\{e_i \otimes f_i\}$ and $\{e'_i \otimes f'_i\}$ for the domain and codomain respectively. The matrix $S \otimes T$ has the form

$$S \otimes T = \begin{pmatrix} s_{11}T & s_{12}T & \cdots \\ s_{21}T & s_{22}T & \cdots \\ & \cdots & \end{pmatrix}.$$

Example 1.2.13. Consider the matrices S and T:

$$S = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} \qquad T = \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \end{pmatrix}.$$

Then the tensor product of $S \otimes T$ is

$$\begin{aligned} S \otimes T &= \begin{pmatrix} s_{11}T & s_{12}T \\ s_{21}T & s_{22}T \end{pmatrix} \\ &= \begin{pmatrix} s_{11}t_{11} & s_{11}t_{12} & s_{11}t_{13} & s_{12}t_{11} & s_{12}t_{12} & s_{12}t_{13} \\ s_{11}t_{21} & s_{11}t_{22} & s_{11}t_{23} & s_{12}t_{21} & s_{12}t_{22} & s_{12}t_{23} \\ s_{21}t_{11} & s_{21}t_{12} & s_{21}t_{13} & s_{22}t_{11} & s_{22}t_{12} & s_{22}t_{13} \\ s_{21}t_{21} & s_{21}t_{22} & s_{21}t_{23} & s_{22}t_{21} & s_{22}t_{22} & s_{22}t_{23} \end{pmatrix}. \end{aligned}$$

Remark 1.2.14. The adjacency matrix of $P_m \square P_n$ is

$$B(P_m) \otimes I_n + (I_m \otimes B(P_n)),$$

where $\otimes$ is the tensor product of matrices.

1.3 Proof of Theorem 1.1.1

As established in Proposition 1.2.10, $\operatorname{perm}(A)$ is $T_C(x, y)$. But the permanent is hard to compute, so we modify A in Definition 1.2.7 to get a weighted matrix A'. We now need to show that $\det(A')$ and $\operatorname{perm}(A)$ differ by a constant factor.

Roughly, the idea is as follows. Any nonzero term in the permutation expansion of $\det(A')$ corresponds to a tiling of C. Let the permutations σ and τ correspond to two tilings T_σ and T_τ. When we place one on top of another, $T_\sigma \cup T_\tau$ is a disjoint union of cycles. One can transform T_σ to T_τ by simply rotating a single cycle at a time. It suffices to show that if T_τ is obtained by T_σ by rotating a single cycle, then the coefficient for their corresponding monomials in $\det(A')$ have the same value α. The details begin with the following lemma.

Lemma 1.3.1. *For any weighted closed even cycle* $\{v_1, \cdots, v_{2k-1}, v_{2k} = v_1\}$ *in* G_C,

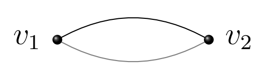
$$\pi = \left.\frac{\prod_{i\ odd} wt(v_i, v_{i+1})}{\prod_{i\ even} wt(v_i, v_{i+1})}\right|_{x=y=1} = (-1)^{k+\ell-1},$$

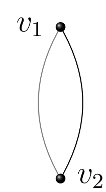
where ℓ *is the number of integer points strictly enclosed in the cycle.*

The proof we give next is based on an online lecture note by Levine [6].

Proof. We prove the lemma by an induction on the area enclosed by the cycle.

The base case is when the area enclosed is 0,





In the left case, we have a 2-cycle,

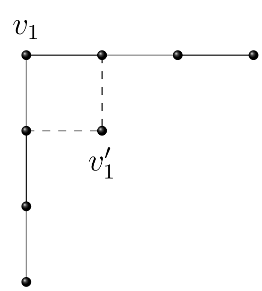
$$\left.\frac{\mathrm{wt}(v_1, v_2)}{\mathrm{wt}(v_2, v_1)}\right|_{x=y=1} = \left.\frac{y}{y}\right|_{x=y=1} = 1 = (-1)^{1+0-1}.$$

Similarly, in the right case,

$$\left.\frac{\mathrm{wt}(v_1, v_2)}{\mathrm{wt}(v_2, v_1)}\right|_{x=y=1} = \left.\frac{ix}{ix}\right|_{x=y=1} = 1 = (-1)^{1+0-1}.$$

In the induction step, without loss of generality, we can assume that v_1 is the vertex in the leftmost column. We consider the following three cases.

Case 1.

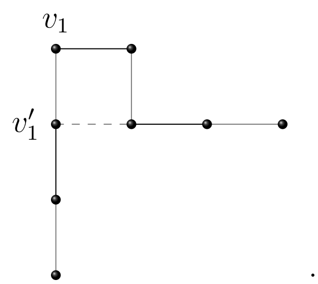


.

This is the case in which it is possible to remove v_1 and replace the two edges connected to v_1 with the two dotted edges, as shown above. The new starting vertex is v_1'. This decreases the area by 1 unit. The new variables are $k' = k$, $\ell' = \ell - 1$. Hence the new ratio π' is, using the inductive hypothesis,

$$\pi = \pi' \frac{y^2}{(ix)^2}\bigg|_{x=y=1} = -(-1)^{\ell'+k'-1} = (-1)^{k+\ell-1}.$$

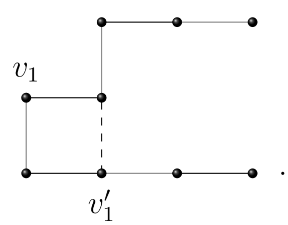
Case 2.



Here, it is possible to remove v_1 and replace the three edges connecting to it by a single horizontal edge, as shown above. Similarly, we have new variables $k' = k - 1$, $\ell' = \ell$. The new ratio is

$$\pi = \frac{y^2}{(ix)^2}\frac{1}{\pi'}\bigg|_{x=y=1} = -(-1)^{\ell'+k'-1} = (-1)^{\ell+k-1}.$$

Case 3. The third case is obtained by rotating the second case:



As an analogy of case 2, the induction step stands for case 3. □

Proposition 1.3.2.

$$\det(A') = \alpha \operatorname{perm}(A) = \alpha T_C(x, y),$$

where $\alpha \in \{1, -1, i, -i\}$.

Proof. We have,

$$\det(A') = \sum_{\sigma \in S_d} \operatorname{sgn}(\sigma) \prod_{i=1}^{d} \operatorname{wt}(w_i, b_{\sigma(i)}).$$

A term in the above sum is nonzero if and only if σ corresponds to a tiling T_σ of the checkerboard C, and in that case we write

$$\operatorname{sgn}(\sigma)\prod_{i=1}^{d}\operatorname{wt}(w_i, b_{\sigma(i)}) = \operatorname{sgn}(\sigma)(i)^{v_\sigma}x^{v_\sigma}y^{h_\sigma},$$

where v_σ and h_σ are the number of vertical and horizontal tiles in T_σ. It now suffices to show that for any pair of permutations σ, $\tau \in S_d$ corresponding to valid tilings, we have

$$\operatorname{sgn}(\sigma)(i)^{v_\sigma} = \operatorname{sgn}(\tau)(i)^{v_\tau} =: \alpha. \tag{1.7}$$

We first argue that we may assume σ is the identity permutation. Suppose T_σ has edges $\{(w_i, w_{\sigma(i)})\}$ and T_τ has edges $\{(w_i, w_{\tau(i)})\}$. Relabel the black tiles by $b_i \mapsto b_{\sigma^{-1}(i)}$. Then T_σ under this relabeling corresponds to the identity permutation $\sigma' = \mathrm{id}$, and is denoted by $T_{\sigma'} = T_{\mathrm{id}}$. Similarly, under this relabeling, T_τ corresponds to the permutation $\tau' = \sigma^{-1}\tau$, and is denoted by $T_{\tau'}$. Since the number of vertical tiles in a tiling does not depend on the labeling of the black vertices, then

$$\begin{aligned}
\operatorname{sgn}(\sigma)(i)^{v_\sigma} = \operatorname{sgn}(\tau)(i)^{v_\tau} &\iff \operatorname{sgn}(\sigma)(i)^{v_{\mathrm{id}}} = \operatorname{sgn}(\tau)(i)^{v_{\tau'}}\\
&\iff (i)^{v_{\mathrm{id}}} = \operatorname{sgn}(\sigma^{-1}\tau)(i)^{v_{\tau'}}\\
&\iff (i)^{v_{\sigma'}} = \operatorname{sgn}(\sigma')(i)^{v_{\sigma'}} = \operatorname{sgn}(\tau')(i)^{v_{\tau'}}
\end{aligned}$$

Thus, we may assume σ is the identity permutation.

Next, assume that we have relabelled the black vertices so that $T_{\mathrm{id}} = T_\sigma$ is a valid tiling, and let T_τ be any valid tiling. Place T_{id} and T_τ on top of each other. Then, $T_{\mathrm{id}} \cup T_\tau$ is a disjoint union of cycles in G_C, and we can transform T_{id} to T_τ by rotating one cycle at a time. Hence, we may assume that $T_{\mathrm{id}} \cup T_\tau$ contains a single cycle of length k for some $k \in \mathbb{Z}$. It follows that τ' contains a cyclic permutation cycle of length k.

By Lemma 1.3.1,

$$\begin{aligned}
\left.\frac{\prod_{i=1}^{d}\operatorname{wt}(w_i, b_{\mathrm{id}})}{\prod_{i=1}^{d}\operatorname{wt}(w_i, b_{\tau(i)})}\right|_{x=y=1} &= \left.\frac{\prod_{i=1}^{k}\operatorname{wt}(w_i, b_{\mathrm{id}})}{\prod_{i=1}^{k}\operatorname{wt}(w_i, b_{\tau(i)})}\right|_{x=y=1}\\
&= \left.\frac{\operatorname{wt}(w_1, b_1)\operatorname{wt}(w_2, b_2)\cdots\operatorname{wt}(w_k, b_k)}{\operatorname{wt}(w_1, b_{\tau(1)})\operatorname{wt}(w_2, b_{\tau(2)})\cdots\operatorname{wt}(w_k, b_{\tau k})}\right|_{x=y=1}\\
&= (-1)^{k+\ell-1},
\end{aligned}$$

where ℓ is the number of vertices enclosed by $T_{\mathrm{id}} \cup T_\tau$. Since the interior of $T_{\mathrm{id}} \cup T_\tau$ is a disjoint union of even cycles, ℓ is even. So,

$$\left.\frac{\operatorname{sgn}(\mathrm{id})\prod_{i=1}^{d}\operatorname{wt}(w_i, b_{\mathrm{id}})}{\operatorname{sgn}(\tau)\prod_{i=1}^{d}\operatorname{wt}(w_i, b_{\tau(i)})}\right|_{x=y=1} = \frac{\operatorname{sgn}(\sigma)}{\operatorname{sgn}(\tau)}\cdot(-1)^{d+\ell-1} = \frac{(-1)^{d+\ell-1}}{(-1)^{d-1}} = 1.$$

On the other hand, since

$$\prod_{i=1}^{d} \mathrm{wt}(w_i, b_i)|_{x=y=1} = (i)^{v_{\mathrm{id}}}$$
$$\prod_{i=1}^{d} \mathrm{wt}(w_i, b_{\tau(i)})|_{x=y=1} = (i)^{v_\tau},$$

Equation (1.7) follows. □

Now we can prove Theorem 1.1.1.

Proof of Theorem 1.1.1. A conclusion from Proposition 1.3.2 is that

$$T_C^2(x,y) = \mathrm{Perm}(A)^2 = \frac{1}{\alpha^2}\det(A')^2 = \frac{1}{(\pm i^d)^2}\det(A')^2. \tag{1.8}$$

We can compute $\det(A')$ to get $T_C(x,y)$. It is still difficult to find $\det(A')$, so we can compute it indirectly through tensor products of matrices. Define the *Kasteleyn matrix* K to be

$$K = \begin{pmatrix} 0 & A' \\ A'^t & 0 \end{pmatrix}.$$

Recall the generalized Laplace expansion:

$$\begin{aligned}
\det(K) &= (-1)^{1+2+\cdots+mn/2+mn/2+1+\cdots+mn}\det(A')\det(A'^t) \\
&= (-1)^{2\binom{\frac{mn}{2}+1}{2}+(\frac{mn}{2})^2}\det(A')\det(A'^t) \\
&= (-1)^{mn/2}\det(A')\det(A'^t) \\
&= (-1)^d\det(A')^2.
\end{aligned}$$

In the second to last line, we used the fact that for a symmetric matrix A', $\det(A')$=$\det(A'^t)$. Using Equation (1.8), we can conclude that

$$T_C^2(x,y) = \frac{1}{(\pm i^d)^2}\det(A')^2 = \frac{(-1)^d}{(i^2)^d}\det(K) = \frac{(-1)^d}{(-1)^d}\det(K) = \det(K). \tag{1.9}$$

All we need to do now is to find $\det(K)$. Note that K is the adjacency matrix for graph G. Recall that

$$G = P_m \square P_n,$$

where P_m and P_n are path graphs, and $\square$ is the Cartesian product. Hence, by Remark 1.2.14,

$$K = ixB(P_m) \otimes I_n + y(I_m \otimes B(P_n)). \tag{1.10}$$

Proposition 1.3.3. *The eigenvalues of K are $\lambda_j + i\mu_k$, where λ_j and μ_k are the eigenvalues of $B(P_m)$ and $B(P_n)$ respectively.*

Proof. Let v_j be an eigenvector of $B(P_m)$ and w_k be an eigenvector of $B(P_n)$. By Equation (1.10),

$$\begin{aligned}
K(v_j \otimes w_k) &= (ixB(P_m) \otimes I_n + y(I_m \otimes B(P_n)))(v_j \otimes w_k) \\
&= ixB(P_m)v_j \otimes I_n w_k + y(I_m v_j \otimes B(P_n)w_k) \\
&= ixB(P_m)v_j \otimes w_k + yv_j \otimes B(P_n)w_k \\
&= ix(\lambda_j v_j) \otimes w_k + yv_j \otimes (\mu_k w_k) \\
&= ix\lambda_j(v_j \otimes w_k) + y\mu_k(v_j \otimes w_k) \\
&= (ix\lambda_j + y\mu_k)(v_j \otimes w_k).
\end{aligned}$$

□

Therefore,

$$\begin{aligned}
\det(K) &= \prod_{j=1}^{m}\prod_{k=1}^{n} ix\lambda_j + y\mu_k \\
&= \prod_{j=1}^{m}\prod_{k=1}^{n}\left(2ix\cos\frac{j\pi}{m+1} + 2y\cos\frac{k\pi}{n+1}\right) \\
&= \prod_{j=1}^{\frac{m}{2}}\prod_{k=1}^{n}\left[\left(2ix\cos\frac{j\pi}{m+1} + 2y\cos\frac{k\pi}{n+1}\right)\left(-2ix\cos\frac{j\pi}{m+1} + 2y\cos\frac{k\pi}{n+1}\right)\right] \\
&= \prod_{j=1}^{\frac{m}{2}}\prod_{k=1}^{n}\left(4x^2\cos^2\frac{j\pi}{m+1} + 4y^2\cos^2\frac{k\pi}{n+1}\right).
\end{aligned}$$

Thus, by Equation (1.9),

$$T_C^2(x,y) = \prod_{j=1}^{\frac{m}{2}}\prod_{k=1}^{n}\left(4x^2\cos^2\frac{j\pi}{m+1} + 4y^2\cos^2\frac{k\pi}{n+1}\right)$$

and, therefore,

$$T_C(x,y) = \prod_{j=1}^{m}\prod_{k=1}^{n}\left(4x^2\cos^2\frac{j\pi}{m+1} + 4y^2\cos^2\frac{k\pi}{n+1}\right)^{\frac{1}{4}}.$$

□

Corollary 1.3.4. *Let C be a $2 \times n$ checkerboard, with n arbitrary. Then*

$$F_{n+1}(x,y) = \prod_{h=1}^{\lfloor \frac{n}{2} \rfloor} \left(x^2 + 4y^2 \cos^2 \left(\frac{h\pi}{n+1} \right) \right), \tag{1.11}$$

where $F_{n+1}(x,y)$ is the Fibonacci polynomial, defined as follows:

$$\begin{aligned} F_0(x,y) &= 0 \\ F_1(x,y) &= 1 \\ F_{n+1}(x,y) &= x \cdot F_n(x,y) + y^2 \cdot F_{n-1}(x,y), \end{aligned}$$

for $n \geq 1$.

Proof. The characteristic polynomial of $\{F_n(x,y)\}$ is

$$P(z) = z^2 - xz - y^2,$$

which has roots:

$$\begin{aligned} \alpha &= \frac{x + \sqrt{x^2+4y^2}}{2} \\ \beta &= \frac{x - \sqrt{x^2+4y^2}}{2}. \end{aligned}$$

Note that here we are regarding $F(x,y)$ as a function from $\mathbb{C}^2 \to \mathbb{C}^2$. The theory of linear recurrence says that $F_n(x,y) = a\alpha^n + b\beta^n$ for some $a, b \in \mathbb{C}$. To determine a and b, we use the initial conditions of the recurrence:

$$\begin{cases} 0 & F_0 = a \cdot \alpha^0 + b \cdot \beta^0 = a + b \\ 1 & F_1 = a \cdot \alpha + b \cdot \beta. \end{cases}$$

Solving the above linear equations, we get

$$a = \frac{1}{\alpha - \beta} = -b.$$

Therefore, simple algebra gives $F_n(x,y) = \frac{\alpha^n - \beta^n}{\alpha - \beta}$.

We now claim that $x = 2iy \cos \frac{k\pi}{n}$ is a root of $F_n(x,y)$ for $k = 1, 2, \cdots, n-1$. Let

$\theta = \frac{k\pi}{n}$, for $k = 1, 2, \cdots, n-1$. Using the previous expression, see that

$$\begin{aligned}
F_n(2iy\cos\theta, y) &= \frac{(iy\cos\theta + y\sin\theta)^n - (iy\cos\theta - y\sin\theta)^n}{2y\sin\theta} \\
&= \frac{(i)^n y^n((\cos\theta - i\sin\theta)^n - (\cos\theta + i\sin\theta)^n)}{2y\sin\theta} \\
&= \frac{(-i)(-i)^{n-1}y^{n-1}(\cos(-\theta n) + i\sin(-\theta n) - \cos(\theta n) - i\sin(\theta n))}{2\sin\theta} \\
&= \frac{(-i)^{n-1}y^{n-1}(2\sin(\theta n))}{2\sin\theta} \\
&= \frac{(-i)^{n-1}y^{n-1}\sin(\theta n)}{\sin\theta},
\end{aligned}$$

which is 0. Since $x = 2iy\cos\theta$ is a root for $F_n(x, y)$, we can decompose:

$$F_n(x, y) = \prod_{k=1}^{n-1}\left(x - 2iy\cos\frac{k\pi}{n}\right) = \prod_{h=1}^{\lfloor\frac{n-1}{2}\rfloor}\left(x^2 + 4y^2\cos^2\left(\frac{h\pi}{n}\right)\right).$$

Then, $F_{n+1}(x, y)$ would be

$$F_{n+1}(x, y) = \prod_{h=1}^{\lfloor\frac{n}{2}\rfloor}\left(x^2 + 4y^2\cos^2\left(\frac{h\pi}{n+1}\right)\right),$$

which is the exact result we get from $T_{2\times n}(x, y)$, by Kasteleyn's formula in Theorem 1.1.1. $\square$

Chapter 2

A Combinatorial Idea Using Complex Weights

2.1 Combinatorial Ideas in Tiling Problems

In Chapter 1, we proved Kasteleyn's result using a mixture of algebra and combinatorics. We described a bijection between perfect matchings in a graph and tilings in a checkerboard, and the number of perfect matchings is the permanent of the adjacency matrix. The crucial combinatorics in the proof comes in the proof that the weighted determinant is equal to the permanent of the adjacency matrix. Using some algebra, we found the eigenvalues and eigenvectors of a matrix associated with the adjacency matrix, and thus we got the determinant.

Various tiling problems require different mixtures of algebra and combinatorics to solve. Our proof of Corollary 1.3.4 was primarily algebraic, relying on a special case of Kasteleyn's theorem. Alternatively, we now give a proof that is primarily combinatorial.

Theorem 2.1.1. *Let $T_{2\times n}(x, y)$ be the tiling generating function as in Chapter 1:*

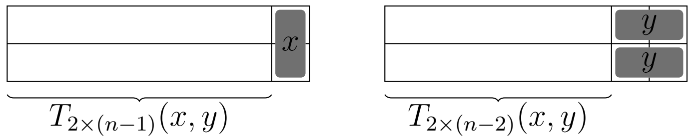
$$T_C(x, y) = \sum_{h,v} t(h, v)x^v y^h.$$

Then, $T_{2\times n}(x, y)$ is the Fibonacci polynomial $F_{n+1}(x, y)$.

Proof. For any $2 \times n$ checkerboard, we can just look at its right most end:

$2 \times n$

If the last tile is placed vertically, then the generating function is $x \cdot T_{2\times(n-1)}$; if the last tile is horizontal, then the generating function is $y^2 \cdot T_{2\times(n-2)}$:



So $T_{2\times n}(x,y)$ has the recurrence:

$$
\begin{aligned}
T_{2\times 1}(x,y) &= x \\
T_{2\times 2}(x,y) &= x^2 + y^2 \\
T_{2\times n}(x,y) &= x \cdot T_{2\times(n-1)} + y^2 \cdot T_{2\times(n-2)}.
\end{aligned}
$$

Since $T_{2\times n}(x,y)$ has the same recurrence as the Fibonacci polynomial $F_{n+1}(x,y)$, and $T_{2\times 1}(x,y) = F_2(x,y)$, $T_{2\times 2}(x,y) = F_3(x,y)$, we conclude that

$$T_{2\times n}(x,y) = F_{n+1}(x,y).$$

□

In fact, we can now use Theorem 2.1.1 to give an independent proof of the $2 \times n$ case of Kasteleyn's formula.

Corollary 2.1.2. *The tiling generating function is*

$$T_{2\times n}(x,y) = \prod_{h=1}^{\lfloor \frac{n}{2} \rfloor} \left(x^2 + 4y^2 \cos^2\left(\frac{h\pi}{n}\right)\right).$$

Proof. By Theorem 2.1.1, we have $T_{2\times n}(x,y) = F_{n+1}(x,y)$. The result follows by the proof of Corollary 1.3.4. □

In Chen's undergrad thesis [3], Benjamin and Chen seek a proof of Kasetelyn's formula in the case $2m \times 2n$ using a new combinatorial idea, which we present in the next section. Chen's thesis makes some progress but does not succeed in finding this new proof, even in the $2 \times 2n$ case. Our thesis began as an attempt to complete the proof in this case.

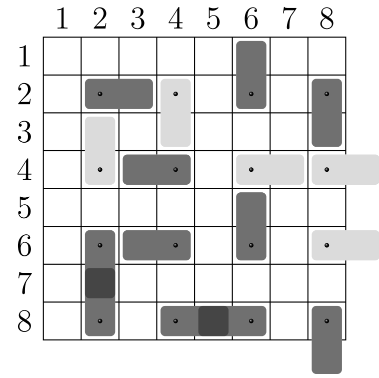


Figure 2.1: A pseudo-tiling of a 8×8 checkerboard.

2.2 A Combinatorial Idea for Kasteleyn's Formula

Using the fact that

$$2\cos\theta = e^{i\theta} + e^{-i\theta},$$

we can rewrite Kasteleyn's formula in Theorem 1.1.1. The number of tilings of a $2m \times 2n$ checkerboard is

$$\prod_{j=1}^{m}\prod_{k=1}^{n}\left[1+1+1+1+\omega_{2m+1}^{j}+\omega_{2m+1}^{-j}+\omega_{2n+1}^{k}+\omega_{2n+1}^{-k}\right] \tag{2.1}$$

where ω_r is a primitive r^{th} root of unity.

Expanding Equation (2.1) gives us all the possible products of chosen terms in the sum. This is indicating a way to weight tiles and look at the sum of all possible weights. We will set up the weights and tilings in this section.

Definition 2.2.1. Number the rows of a $2m \times 2n$ checkerboard from top to bottom, and columns from left to right. A *pseudo-domino* is a white or black domino covering a $(2j, 2k)$ square. It can have four directions: up, down, left or right. A *pseudo-tiling* is a tiling comprised of pseudo-dominoes.

Example 2.2.2. See Figure 2.1 for an example of a pseudo-tiling of an 8×8 checkerboard.

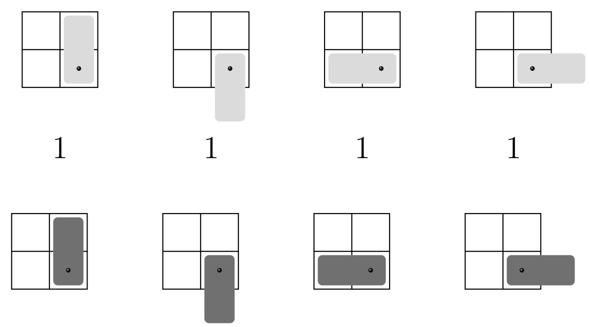
Theorem 2.2.3 (Benjamin, Tucker, Klawe [3]). *A pseudo-tiling has either two pseudo-dominos overlapping, or some pseudo-domino hanging over the edge, or a cycle of pseudo-dominos enclosing an odd region, or there is a unique way to extend the cover to a standard tiling of the entire board.*

Definition 2.2.4. *A proper pseudo-tiling* is an all-white pseudo-tiling that can extend to a standard tiling of the checkerboard. Otherwise, it is *improper*. A pseudo-domino in a pseudo-tiling is *proper* if it is white, does not hang off the edge of the board, and does not overlap with another pseudo-domino. Otherwise, the pseudo-domino is *improper*.

Theorem 2.2.3 and Equation (2.1) suggest that there is a weight function that could lead to a combinatorial proof of Kasteleyn's theorem. The idea is that we can weight each pseudo-domino, and the sum of all possible pseudo-tilings weights is Equation (2.1). Since Kasteleyn's theorem says that Equation (2.1) is the number of standard tilings, weights of improper pseudo-tilings cancel, leaving the weights of proper pseudo-tilings.

The eight terms in Equation (2.1) imply that there are 8 possible situations for each pseudo-domino, which matches the eight color and direction choices of each pseudo-domino.

Definition 2.2.5. Let ω_r be the r^{th} root of unity. *The weight of a pseudo-domino* depends on its color and direction:



$$1 \qquad 1 \qquad 1 \qquad 1$$

$$\omega_{2m+1}^{j} \qquad \omega_{2m+1}^{-j} \qquad \omega_{2n+1}^{k} \qquad \omega_{2n+1}^{-k}.$$

The weight of a pseudo-tiling is the product of the weight of its pseudo-dominoes. *The weight of a set of pseudo-tilings* is the sum of the weight of its elements.

Remark 2.2.6. A proper pseudo-tiling has a weight of 1. Thus, the weights of all proper pseudo-tilings is the number of them.

Example 2.2.7. Using the 8×8 checkerboard in Example 2.2.2, the weight of that specific pseudo-tiling is

$$\begin{aligned} wt(P) &= \omega_9^{-1} \cdot 1 \cdot \omega_9^3 \omega_9^{-4} \cdot 1 \cdot \omega_9^2 \cdot 1 \cdot 1 \cdot \omega_9^{-1} \omega_9^2 \omega_9^3 \cdot 1 \cdot \omega_9^1 \omega_9^{-2} \omega_9^3 \omega_9^{-4} \\ &= \omega_9^2. \end{aligned}$$

Important Idea. By what we have explained so far, we see Kasteleyn's formula is equivalent to having the sum of the weights of all improper pseudo-tilings be 0. Thus, if we can prove that the weights of all improper pseudo-tilings is 0 independent of Kasteleyn's formula, we get a new proof of Kasteleyn's formula. We would hope that such a proof would be largely combinatorial rather than algebraic.

Chen's thesis begins with this combinatorial idea and has some results.

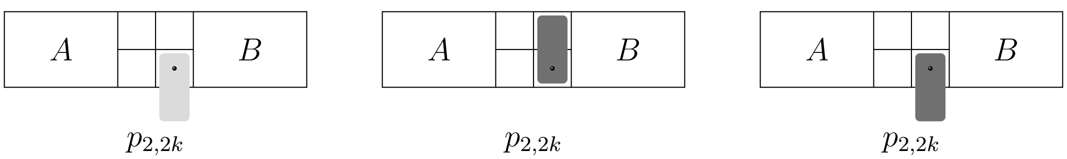
Consider a pseudo-tiling of $2 \times 2n$ with at least one improper vertical pseudo-domino, i.e., some black vertical pseudo-dominoes or a white down-facing pseudo-domino. We can modify that specific improper vertical pseudo-domino to create a set that has a vanishing weight.

Proposition 2.2.8 (Benjamin, Chen, 2010)**.** *For a $2 \times 2n$ checkerboard, the total weight of all pseudo-tilings with improper vertical pseudo-dominoes is 0.*

Proof. Let a pseudo-tiling P have some vertical improper pseudo-dominoes. Let $(2, 2k)$ be the first position of an improper vertical pseudo-domino. We write

$$P = [A\, p_{2,2k}\, B],$$

where A is a pseudo-tiling with no vertical improper pseudo-dominos, $p_{2,2k}$ is the first improper vertical pseudo-domino, and B can be any pseudo-tiling. Then such a pseudo-tiling has three options at position $(2, 2k)$: $p_{2,2k}$ can either be a white down, a black up or a black down.



Let $\Psi_{A,p_{2,2k},B}$ be the set that contains the three pseudo-tilings above. Then the weight of $\Psi_{A,p_{2,2k},B}$ is

$$\begin{aligned}
\mathrm{wt}(\Psi_{A,p_{2,2k},B}) &= \mathrm{wt}(A) \cdot 1 \cdot \mathrm{wt}(B) + \mathrm{wt}(A) \cdot \omega_3 \cdot \mathrm{wt}(B) + \mathrm{wt}(A) \cdot \omega_3^2 \cdot \mathrm{wt}(B) \\
&= \mathrm{wt}(A) \cdot \mathrm{wt}(B)(1 + \omega_3 + \omega_3^2) \\
&= \mathrm{wt}(A) \cdot \mathrm{wt}(B) \cdot 0 \\
&= 0.
\end{aligned}$$

Therefore, the weight of all such $\Psi_{A,p_{2,2k},B}$, i.e., the weight of the set of all pseudo-tilings with at least one improper vertical domino, Ψ, is

$$\sum_{A,p_{2,2k},B} \mathrm{wt}(\Psi_{A,p_{2,2k},B}) = \mathrm{wt}(\Psi) = 0.$$

□

Proposition 2.2.8 tells us that the set of all pseudo-tilings with at least one improper vertical pseudo-domino has weight 0. Now we concentrate on pseudo-tilings with no improper vertical pseudo-dominoes. Thus, to prove Kasteleyn's formula in the $2 \times 2n$ case, it suffices to show that the set of all pseudo-tilings with no improper vertical pseudo-dominoes has weight 0. This thesis is primarily inspired by this question.

Chapter 3

Chebyshev Polynomial and Complex Weight

From now on, we will only consider pseudo-tilings of a $2 \times 2n$ checkerboard.

3.1 Pseudo-tilings with No Vertical Pseudo-dominoes

In the previous chapter, we partitioned the set of all pseudo-tilings with at least one improper vertical domino Ψ into subsets $\Psi_{A,p_{2,2k},B}$ of size 3. Each subset $\Psi_{A,p_{2,2k},B}$ has weight 0, hence the weight of Ψ is 0. Now, we want to do something similar with the remaining set of pseudo-tilings.

Let X_n be the set of *improper* pseudo-tilings of a $2 \times 2n$ checkerboard with no improper vertical pseudo-dominoes. Let $X_{n,h}$ be the subset of X_n with h horizontal pseudo-dominoes. Then $X_n = \bigsqcup_h X_{n,h}$. The main goal of this chapter is to give an algebraic proof of the following theorem.

Theorem 3.1.1. *For arbitrary n and h,*

$$\mathrm{wt}(X_{n,h}) = 0.$$

3.2 Chebyshev Polynomial of the Third Kind

We now turn to the seemingly unrelated topic of Chebyshev polynomials of the third kind. In Theorem 3.2.2, rewriting this polynomial gives a generating function of weights of pseudo-tilings. We first introduce some basics about the Chebyshev polynomial, which come from Mason and Handscomb [7].

Definition 3.2.1. *The Chebyshev polynomial of the third kind* $\{V_n\}$ is defined by the following recursion,

$$\begin{aligned} V_0(x) &= 1 \\ V_1(x) &= 2x - 1 \\ V_{n+1}(x) &= 2xV_n(x) - V_{n-1}(x) \end{aligned}$$

for $n \geq 1$.

Rearranging the above equation, we can get another recursion:

$$2xV_n(x) = V_{n+1}(x) + V_{n-1}(x). \tag{3.1}$$

Chebyshev polynomials of the third kind also have interesting relations to trigonometry. Let $x = \cos(\theta)$. Then

$$V_n(\cos(\theta)) = \frac{\cos\left(n + \frac{1}{2}\right)\theta}{\cos\left(\frac{\theta}{2}\right)}.$$

Therefore, the zeros of $V_n(x)$ are

$$\cos\left(\frac{2k-1}{2n+1}\pi\right) = \cos\left(k\theta - \frac{\pi}{2n+1}\right)$$

where $\theta = \frac{2\pi}{2n+1}$ for $k = 1, \ldots, n$.

From Equation (3.1), it follows that the leading coefficient of $V_n(x)$ is 2^n. Therefore, $V_n(x)$ can be factored as

$$V_n(x) = 2^n \prod_{k=1}^{n} \left(x - \cos\left(k\theta - \frac{\pi}{2n+1}\right)\right) = \prod_{k=1}^{n} \left(2x - 2\cos\left(k\theta - \frac{\pi}{2n+1}\right)\right). \tag{3.2}$$

The next theorem shows that $V_n(x)$ is a generating function of weights of pseudo-tilings with only horizontal pseudo-dominoes.

Theorem 3.2.2. *Let* $\omega = 2\pi/(2n+1)$. *Then*

$$V_n(x) = \prod_{k=1}^{n} \left(2x + \omega^k + \omega^{-k}\right) = \sum_{w=1}^{n} g_{n,w} x^w,$$

where $g_{n,w}$ *is the sum of weights of* $2 \times 2n$ *pseudo-tilings with only horizontal pseudo-dominoes and* w *white pseudo-dominoes.*

Proof. Starting from the factorization of $V_n(x)$ in Equation (3.2),

$$\begin{aligned}
V_n(x) &= 2^n \prod_{k=1}^{n} \left(x - \cos\left(k\theta - \frac{\pi}{2n+1} \right) \right) \\
&= 2^n \prod_{k=1}^{n} \left(x + \cos\left(\pi - k\theta + \frac{\pi}{2n+1} \right) \right) \\
&= 2^n \prod_{k=1}^{n} \left(x + \cos\left(\frac{(n-k+1)2\pi}{2n+1} \right) \right) \\
&= 2^n \prod_{k=1}^{n} \left(x + \cos\left(\frac{2k\pi}{2n+1} \right) \right) \\
&= \prod_{k=1}^{n} \left(2x + 2\cos\left(\frac{2k\pi}{2n+1} \right) \right) \\
&= \prod_{k=1}^{n} \left(2x + \omega^k + \omega^{-k} \right).
\end{aligned}$$

Let us return to Equation (2.1):

$$\prod_{j=1}^{m} \prod_{k=1}^{n} \left[1 + 1 + 1 + 1 + \omega_{2m+1}^{j} + \omega_{2m+1}^{-j} + \omega_{2n+1}^{k} + \omega_{2n+1}^{-k} \right].$$

If we take out all summands corresponding to vertical pseudo-dominoes, we have a formula for the weights of all pseudo-tilings with only horizontal pseudo-dominoes:

$$\prod_{k=1}^{n} \left[1 + 1 + \omega_{2n+1}^{k} + \omega_{2n+1}^{-k} \right]. \tag{3.3}$$

The two 1's correspond to left and right white pseudo-dominoes. Replacing the 1's with x's, Equation (3.3) is exactly $V_n(x)$:

$$\begin{aligned}
V_n(x) &= \prod_{k=1}^{n} \left(2x + \omega^k + \omega^{-k} \right) \\
&= \sum_{w=1}^{n} g_{n,w} x^w.
\end{aligned}$$

□

3.3 Generating Functions

Let Y_n be the set of all $2 \times 2n$ *proper* pseudo-tilings, and let $Y_{n,h}$ be the subset of Y_n with exactly h horizontal pseudo-dominoes. Thus Y_n is the disjoint union of $Y_{n,h}$. The set of all pseudo-tilings, both proper and improper, with exactly h horizontal pseudo-dominoes, is $X_{n,h} \bigsqcup Y_{n,h}$. In the last section, we found the generating function for weights of $X_{n,n} \bigsqcup Y_{n,n}$. We need to find a general expression for generating functions of weights of all kinds of pseudo-tilings.

Definition 3.3.1. Define the generating function $G_n(h, x)$ to be,

$$G_n(h, x) = \sum_{w=0}^{h} g_{h,w} x^w$$

where $g_{h,w}$ is the sums of weights of $2 \times 2n$ pseudo-tilings with w horizontal white pseudo-dominoes and h total horizontal pseudo-dominoes, with no vertical improper pseudo-dominoes. Thus,

$$G_n(h, 1) = \sum_{w=0}^{h} g_{h,w} = \mathrm{wt}(X_{n,h} \bigsqcup Y_{n,h}).$$

Example 3.3.2. By Theorem 3.2.2, $V_n(x)$ is the generating function for pseudo-tilings of a $2 \times 2n$ checkerboard with all horizontal pseudo-dominoes.

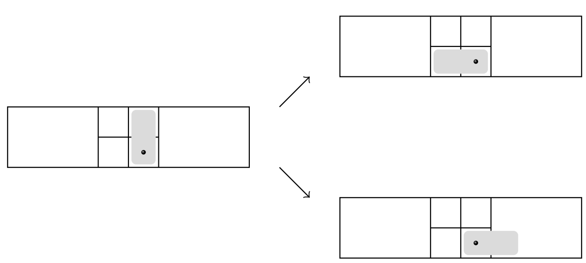
$$V_n(x) = G_n(n, x) = \sum_{w=0}^{n} g_{n,w} x^w.$$

Our next goal is to express $G_n(h, x)$ in terms of $V_n(x) = G_n(n, x)$.

Proposition 3.3.3. *Recall that $g_{h,w}$ is the weight of all $2 \times 2n$ pseudo-tilings with h horizontal pseudo-dominoes and w horizontal white ones. Then,*

$$2((n - h) + 1) \cdot g_{h-1,w-1} = w \cdot g_{h,w}.$$

Proof. Let P be a pseudo-tiling counted by $g_{h-1,w-1}$. Then P has $n - h + 1$ vertical white pseudo-dominoes. Each of these vertical pseudo-dominoes gives rise to two pseudo-tilings counted by $g_{h,w}$:


.

Hence from each such P we form $2 \cdot ((n-h)+1)$ pseudo-tilings counted by $g_{h,w}$. On the other hand, if Q is a pseudo-tiling counted by $g_{h,w}$, then Q gives rise to w pseudo-tilings P counted by $g_{h-1,w-1}$. We find these P by choosing any of the w horizontal pseudo-dominoes in Q and making it vertical. The result follows. □

Theorem 3.3.4. *Let $G_n(h,x)$ be as defined in Definition 3.3.1. Then, $G_n(h-1,x)$ is related to the derivative of $G_n(h,x)$ by the following equation:*

$$G_n'(h,x) = 2(n-h+1)G_n(h-1,x).$$

Proof. If we take the derivative of $G_n(h,x)$, we find

$$\begin{aligned} G_n'(h,x) &= \sum_{w=0}^{h} w \cdot g_{h,w} x^{w-1} \\ &= \sum_{w=0}^{h} 2(n-h+1) \cdot g_{h-1,w-1} x^{w-1} \\ &= 2(n-h+1) \cdot G_n(h-1,x). \end{aligned}$$

The second line is due to Proposition 3.3.3. □

Corollary 3.3.5. *$V_n(x)$ encodes all information of $G_n(h,x)$ for arbitrary n and h:*

$$G_n(h,x) = \frac{V_n^{(n-h)}(x)}{2^{n-h}(n-h)!}.$$

Proof. Using Theorem 3.3.4,

$$\begin{aligned} V_n(x) &= G_n(n,x) \\ V_n'(x) &= G_n'(n,x) = 2G_n(n-1,x) \\ V_n''(x) &= G_n''(n,x) = 2^2 \cdot 2G_n(n-2,x) \\ V_n'''(x) &= G_n'''(n,x) = 2^3 \cdot 2 \cdot 3G_n(n-3,x) \end{aligned}$$

and so on. By induction, if we have $\ell = n-h$ vertical pseudo-dominoes,

$$V_n^{(\ell)}(x) = 2^\ell \cdot \ell! \cdot G_n(n-\ell,x),$$

and the result follows. □

3.4 Proper Pseudo-tilings

In this section, we count the number of proper pseudo-tilings with h horizontal pseudo-dominoes, and relate it to a certain kind of Dyck path.

Proposition 3.4.1. *The number of proper pseudo-tilings in a $2 \times 2n$ checkerboard with h horizontal pseudo-dominoes is $|Y_{n,h}| = \binom{2n-h}{h}$.*

We give two proofs.

Proof. Let V stand for a vertical pseudo-domino, L stand for a left, and R stand for a right pseudo-domino. For a $2 \times 2n$ pseudo-tiling to be proper with h horizontal tiles, we first fix the positions of $n - h$ vertical pseudo-tilings. Then, we place h horizontal pseudo-dominoes in between the vertical pseudo-dominoes. Below is an example of such proper pseudo-tiling of a $2 \times (2 \cdot 14)$ checkerboard,

$$L \;\; L \;\; R \;\; V \;\; L \;\; L \;\; V \;\; R \;\; R \;\; V \;\; L \;\; L \;\; L \;\; L$$

The horizontal pseudo-dominoes after the last V can only be L's. Also, there can not be any RL combinations in between V's since pseudo-dominoes cannot overlap in a proper pseudo-tiling. Say there are x_i L's and y_i R's before the i^{th} V and there are x_{n-h+1} L's after the last V. We want to count the number of solutions to

$$x_1 + y_1 + x_2 + y_2 + \cdots + x_{(n-h)+1} = h,$$

where $x_i, y_i \in \mathbb{N}$. This is a stars and bars problem in which we place bars between each LR pair and replace each V with a bar. A proper pseudo-tiling of the form

$$L \cdots L \;\; R \cdots R \;\; V \;\; L \cdots L \;\; R \cdots R \;\; V \cdots V \;\; L \cdots$$

becomes

$$\underbrace{L \cdots L}_{x_1} \;|\; \underbrace{R \cdots R}_{y_1} \;|\; \underbrace{L \cdots L}_{x_2} \;|\; \underbrace{R \cdots R}_{y_2} \;|\; \cdots \;|\; \underbrace{L \cdots}_{x_{(n-h)+1}}$$

Therefore, by stars and bars, there are

$$\binom{2n - 2h + 1 + h - 1}{h} = \binom{2n - h}{h}$$

ways to have proper pseudo-tilings with h horizontal pseudo-dominoes. $\square$

Another proof uses Dyck paths.

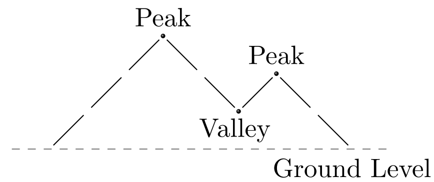


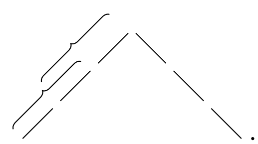
Figure 3.1: Dyck path as a mountain range.

Definition 3.4.2. A *Dyck path* of semilength $n+1$ is a lattice path consisting of moves of the form $(1,1)$ and $(1,-1)$, starting at $(0,0)$, ending at $(2(n+1), 2(n+1))$, and never passing below the x-axis. A $(1,1)$ move is a *rise* and a $(1,-1)$ move is a *fall*.

We think of a Dyck path as a mountain range, see Figure 3.1.

Definition 3.4.3. A Dyck path is *non-decreasing* if the heights of its valleys is a nondecreasing sequence. A *double rise* in a Dyck path is a pair of consecutive rises.

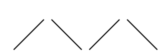
Example 3.4.4. Below is a Dyck path with 2 double rises:



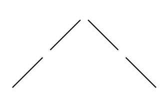
Proposition 3.4.5. *The number of nondecreasing Dyck paths of $n+1$ semilength with h double rises is $|Y_{n,h}| = \binom{2n-h}{h}$.*

Proof. Let $\nu(n,h)$ be the set of such Dyck paths with $n+1$ semilength and h double rises. We proceed by strong induction.

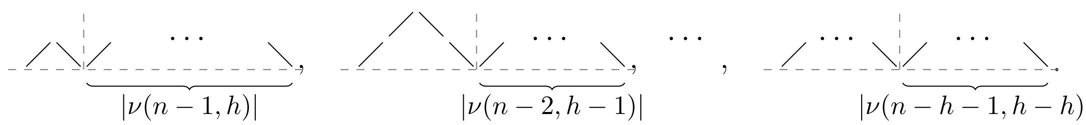
For $n=1$ and $h=0$, there is $\binom{2\times 1-0}{0} = 1$ such nondecreasing Dyck path with $h=0$ double rise



and $\binom{2\times 1-1}{1} = 1$ with $h=1$ double rises



Assume the result holds true for some $n-1$ and $h-1$. Dyck paths in $\nu(n,h)$ can be divided into two disjoint subsets $\nu(n,h) = \nu_1(n,h) \bigsqcup \nu_2(n,h)$, where $\nu_1(n,h)$ has valleys touching the ground level and $\nu_2(n,h)$ does not. To count $|\nu_1(n,h)|$, we can chop off the moves before the first valley that touches the ground level and reduce the problem to a previous case. Note that since the Dyck paths are nondecreasing, if there is some valley touching the ground level, then the first valley has to touch the ground level too. The first valley can occur at quite a few spots on the ground level, so we need to account for each case:

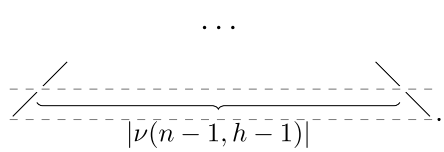


Therefore,

$$|\nu_1(n,h)| = \sum_{i=0}^{h} \binom{2n-h-2-i}{h-i} = \binom{2n-h-1}{h},$$

which is due to the Hockey-stick Identity.

To count $\nu_2(n,h)$, since none of the valleys are touching the ground level, we can take the first and the last move and reduce the counting problem to the previous case of $\nu(n-1,h-1)$. Specifically, doing so gives a bijection between $\nu_2(n,h)$ and $\nu(n-1,h-1)$. The example below explains the bijection:



So we have

$$|\nu_2(n,h)| = |\nu(n-1,h-1)| = \binom{2n-h-1}{h-1}.$$

By the definition of the two subsets, we add them up to get the size of $\nu(n,h)$:

$$|\nu(n,h)| = |\nu_1(n,h)| + |\nu_2(n,h)| = \binom{2n-h-1}{h} + \binom{2n-h-1}{h-1} = \binom{2n-h}{h},$$

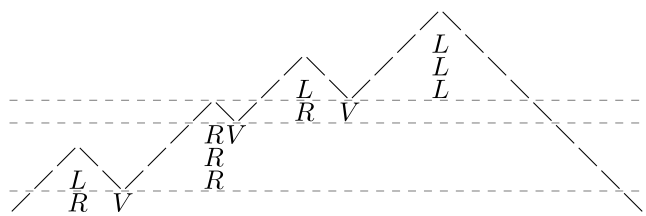
using Pascal's Rule. □

This next proof of Proposition 3.4.1 shows a bijection between proper pseudo-tilings and the Dyck paths in Proposition 3.4.5.

Proof. As laid out in the first proof, for a pseudo-tiling of a $2 \times 2n$ checkerboard with h horizontal pseudo-dominos to be proper, we need to have all L's before R's in between V's. An obvious bijection between Dyck paths in Proposition 3.4.5 and such pseudo-tilings is to count each double rise as a horizontal pseudo-domino and each valley a V. An example of the natural bijection is given below. For a proper pseudo-tiling of $2 \times (2 \cdot 13)$ checkerboard with $h = 10$ horizontal pseudo-dominos

$$L\ R\ V\ R\ R\ R\ V\ L\ R\ V\ L\ L\ L\,,$$

the corresponding Dyck path is as follows,



Since the valleys are non-decreasing, we won't have any RL combination and the horizontal pseudo-dominoes after the last V are all L's. The height of each valley is the number of R's proceeding it. Hence, each such Dyck path matches a proper pseudo-tiling uniquely. We then confirm that this is a bijection, and $|Y_{n,h}| = \binom{2n-h}{h}$. $\square$

3.5 Proof of Theorem 3.1.1

Proposition 3.5.1. *Let $\ell = n - h$. Then*

$$V_n^{(\ell)}(1) = 2^\ell \ell! \binom{n+\ell}{n-\ell} = 2^{n-h}(n-h)!\binom{2n-h}{h}.$$

Proof. We will prove the equality by showing that both sides have the same recurrence.

We first find the recurrence of $V_n^{(\ell)}(x)$. By the recurrence of $V_n(x)$, we have

$$\begin{aligned}
V_{n-1}(x) + V_{n+1}(x) &= 2xV_n(x)\\
V'_{n-1}(x) + V'_{n+1}(x) &= 2V_n(x) + 2xV'_n(x)\\
V''_{n-1}(x) + V''_{n+1}(x) &= 4V'_n(x) + 2xV''_n(x)\\
V'''_{n-1}(x) + V'''_{n+1}(x) &= 6V''_n(x) + 2xV'''_n(x),
\end{aligned}$$

and so on. By induction,

$$V_{n-1}^{(\ell)}(x) + V_{n+1}^{(\ell)}(x) = 2\ell V_n^{(\ell-1)}(x) + 2xV_n^{(\ell)}(x).$$

Setting $x = 1$, we get the following recursion:

$$V_{n+1}^{(\ell)}(1) = 2\ell V_n^{(\ell-1)}(1) + 2V_n^{(\ell)}(1) - V_{n-1}^{(\ell)}(1).$$

Now we look at the recurrence of the right hand side.

$$
\begin{aligned}
2^\ell \ell! \binom{n+1+\ell}{n+1-\ell} &= 2^\ell \ell! \left(\binom{n+\ell-1}{n-\ell+1} + 2\binom{n+\ell}{n-\ell} - \binom{n+\ell-1}{n-\ell-1} \right) \\
&= 2\ell 2^{\ell-1}(\ell-1)! \binom{n+(\ell-1)}{n-(\ell-1)} + 2 \cdot 2^\ell \ell! \binom{n+\ell}{n-\ell} - 2^\ell \ell! \binom{(n-1)+\ell}{(n-1)-\ell}.
\end{aligned}
$$

For the base case, recall from Definition 3.2.1 that $V_0(x) = 1$ and $V_1(x) = 2x - 1$.

$$
\begin{aligned}
V_0(1) &= 2^0 \cdot 0! \cdot \binom{0}{0} = 1 \\
V_0^{(1)}(1) &= 2^1 \cdot 1! \cdot \binom{1}{-1} = 0 \\
V_1(1) &= 2^0 \cdot 0! \cdot \binom{1}{1} = 1.
\end{aligned}
$$

Both sides have the same recursion and thus are equivalent. $\square$

Now we can prove Theorem 3.1.1.

Proof of Theorem 3.1.1. By Proposition 3.5.1,

$$V_n^{(n-h)}(1) = 2^{(n-h)}(n-h)! \binom{2n-h}{h},$$

and thus

$$\frac{V_n^{(n-h)}(1)}{2^{n-h}(n-h)!} = \binom{2n-h}{h}.$$

We can relate this equality to the generating function $G_n(h, x)$ using Corollary 3.3.5:

$$G_n(h, 1) = \binom{2n-h}{h} = |Y_{n,h}|.$$

The second equality is due to Proposition 3.4.5. By Definition 3.3.1,

$$G_n(h, 1) = \mathrm{wt}(X_{n,h} \bigsqcup Y_{n,h}) = \mathrm{wt}(X_{n,h}) + |Y_{n,h}|.$$

The above two equations imply that $\mathrm{wt}(X_{n,h}) = 0$. $\square$

3.6 A New Generating Function

If we inspect the Taylor series of $V_n(x)$, we can find a new generating function for the number of proper pseudo-tilings.

Proposition 3.6.1. *Let ℓ be the number of vertical pseudo-dominoes. Then $V_n(\frac{x}{2}+1)$ is the generating function of the number of proper $2\times 2n$ pseudo-tilings with ℓ vertical pseudo-dominoes.*

Proof. We explore the Taylor series of $V_n(x)$:

$$\begin{aligned}
V_n(x) &= \sum_{w=0}^{n} \frac{V_n^{(w)}(1)}{w!}(x-1)^w \\
&= \sum_{h=0}^{n} \frac{V_n^{(n-h)}(1)}{(n-h)!}(x-1)^{n-h} \\
&= \sum_{h=0}^{h} 2^{n-h}\binom{2n-h}{h}(x-1)^{n-h} \\
&= \sum_{\ell=0}^{n} 2^{\ell}\binom{n+\ell}{n-\ell}(x-1)^{\ell}.
\end{aligned}$$

Thus, we plug in $\frac{x}{2}+1$:

$$\begin{aligned}
V_n\left(\frac{x}{2}+1\right) &= \sum_{\ell=0}^{n} 2^{\ell}\binom{n+\ell}{n-\ell}\left(\frac{x}{2}+1-1\right)^{\ell} \\
&= \sum_{\ell=0}^{n} \binom{n+\ell}{n-\ell}x^{\ell}.
\end{aligned}$$

According to Proposition 3.4.5, the coefficient $\binom{n+\ell}{n-\ell} = \binom{2n-h}{h}$ is $|Y_{n,h}|$. The result follows. □

If we examine the way we extend pseudo-tilings to standard tilings of a $2 \times 2n$ checkerboard, we will find that $V_n(\frac{x}{2}+1)$ is actually a generating function for numbers of standard tilings.

Definition 3.6.2. A *flip* of a domino tiling is a local transform that rotates two dominoes covering a 2×2 square 90 degrees.

Example 3.6.3. Figure 3.2 shows an example of a single flip in a 4×4 checkerboard tiling.

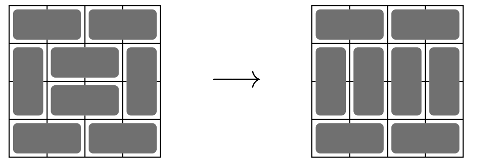


Figure 3.2: A single flip in a 4×4 checkerboard.

Definition 3.6.4. Two domino tilings of a checkerboard are *flip-accessible* if one can be transformed to another by a finite sequence of flips.

Proposition 3.6.5. *A proper $2m \times 2n$ pseudo-tiling P with ℓ vertical and h horizontal pseudo-dominoes can be extended uniquely to a standard tiling with 2ℓ vertical and $2h$ horizontal dominoes.*

Proof. The uniqueness is stated in Theorem 2.2.3. It is well-known by Thurston [12] that any two tilings of a $2m \times n$ checkerboard are flip-accessible. Then if we start with a $2m \times n$ tiling with all vertical dominoes, any other $2m \times n$ tiling can be obtained by finitely many flips. Thus the number of vertical tiles and the number of horizontal tiles in any tiling is even.

Now we number the checkerboard as in Definition 2.2.1 and find pseudo-tilings. Any 2×2 square in the checkerboard has a $(2j, 2k)$ square either on the upper left, upper right or the lower left, lower right corner. To make a pseudo-tiling, we take away the dominoes that do not cover the $(2j, 2k)$ squares. Consequently, we keep a half of the horizontal and a half of the vertical dominoes. □

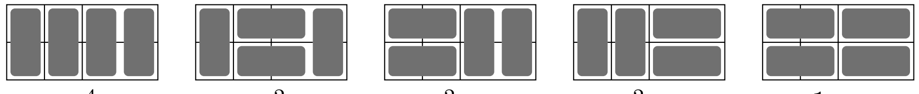
Corollary 3.6.6. *The generating function for the number of standard $2 \times 2n$ tilings by the number of vertical dominoes is $V_n(\frac{x^2}{2} + 1) = \sum_{\ell=0}^{n} \binom{n+\ell}{n-\ell} x^{2\ell}$.*

Proof. By Proposition 3.6.5 and Proposition 3.6.1, the result follows. □

Example 3.6.7. We have

$$V_2\left(\frac{x^2}{2} + 1\right) = x^4 + 3x^2 + 1$$

corresponding to the tilings:



x^4 x^2 x^2 x^2 1

Chapter 4

Combinatorics of Pseudo-tilings

In the last chapter, we checked algebraically that $\mathrm{wt}(X_{n,h}) = 0$. Now we want to develop a combinatorial proof of this equality as Chen did in Proposition 2.2.8. In this chapter, we introduce some of the combinatorial stories we have for Theorem 3.1.1 so far.

We partition the set X_n into a disjoint union of its subsets $X_n = \bigsqcup_h X_{n,h}$. We group pseudo-tilings in $X_{n,h}$ into subsets such that using the identity of roots of unity

$$1 + \omega_{2n+1}^1 + \omega_{2n+1}^2 + \cdots + \omega_{2n+1}^{2n} = 0,$$

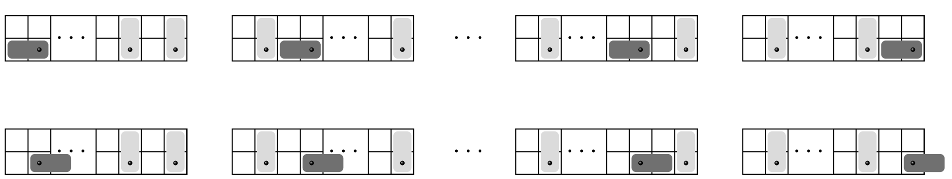
each subset has weight 0. We do this for the cases of $X_{n,1}$, $X_{n,2}$ and $X_{n,n}$.

4.1 Combinatorics Using Set-Grouping

Corollary 4.1.1. *For arbitrary n,*

$$\mathrm{wt}(X_{n,1}) = 0.$$

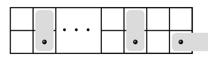
Proof. If $h = 1$, the pseudo-tilings of $X_{n,1}$ that have a black pseudo-domino are



The sum of their weights is

$$\omega_{2n+1}^1 + \omega_{2n+1}^2 + \cdots + \omega_{2n+1}^n + \omega_{2n+1}^{-1} + \omega_{2n+1}^{-2} + \cdots + \omega_{2n+1}^{-n} = -1.$$

The last element of $X_{n,1}$ is the all white pseudo-tiling with a right-facing pseudo-domino at the end:


,

which has a weight of 1. Thus

$$\mathrm{wt}(X_{n,1}) = -1 + 1 = 0.$$

□

Corollary 4.1.2. *For arbitrary n,*

$$\mathrm{wt}(X_{n,2}) = 0.$$

Proof. We can rewrite $X_{n,2}$ as a sum of three disjoint subsets: $X_{n,2} = \bigsqcup_{b=0}^{2} X_{n,2}(b)$, where $X_{n,2}(b) := \{P \in X_{n,2} | P \text{ has } b \text{ black horizontal pseudo-dominoes}\}$. We will split $X_{n,2}(0)$ into two subsets: one with 1 white pseudo-domino hanging over the edge and one with 2 overlapping white pseudo-dominoes. Each one will contribute to the weight of $X_{n,2}(1)$ and $X_{n,2}(2)$, making the total weight of $X_{n,2}$ vanish.

Case of $X_{n,2}(1)$. We follow an idea similar to that in Corollary 4.1.1. There is a surjection between $X_{n,2}(1)$ and $X_{n,1}$. For any pseudo-tiling $P \in X_{n,1}$, we can make one of its vertical pseudo-dominos a horizontal one. Each vertical pseudo-domino will give rise to two such copies since the horizontal pseudo-domino has two possible directions. Such $P \in X_{n,1}$ will make $2(n-1)$ such pseudo-tilings in $X_{n,2}(1)$. See Figure 4.1 for an example. The weight of this set is $2(n-1)\omega_{2n+1}^{j}$ where j depends on the placement of the black horizontal domino. The sum of weights of all pseudo-tilings risen by $X_{n,1}$ is

$$2(n-1)\omega_{2n+1}^{1} + \cdots + 2(n-1)\omega_{2n+1}^{n} + \cdots + 2(n-1)\omega_{2n+1}^{2n} = -2(n-1).$$

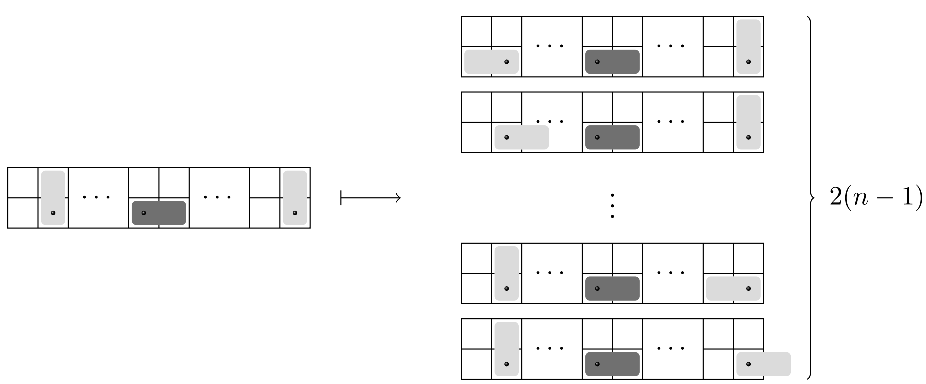


Figure 4.1: The $2(n-1)$ copies of a $P \in X_{n,1}$.

In $X_{n,2}(0)$, there are exactly $2(n-1)$ pseudo-tilings with 1 white pseudo-domino hanging over the edge, because each pseudo-domino, except for the last one, can be a horizontal white pseudo-domino, and each such pseudo-domino can have two directions. The weight of these pseudo-tilings is $2(n-1)$.

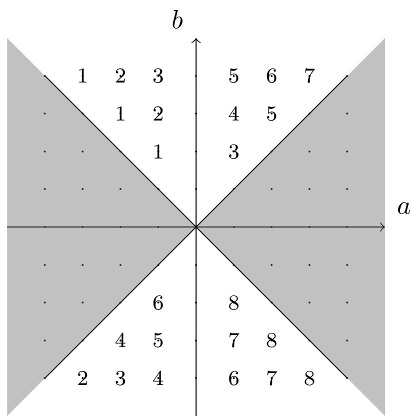
Hence, the weight of all the above pseudo-tilings is

$$-(2n-1)+(2n-1)=0.$$

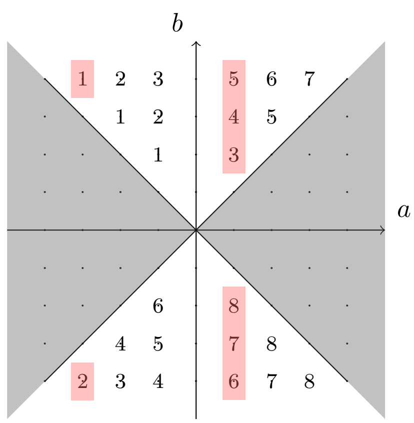
Case of $X_{n,2}(2)$. Let $P \in X_{n,2}(2)$. The weight of P can be written as

$$\mathrm{wt}(P)=\omega_{2n+1}^{a}\cdot\omega_{2n+1}^{b}=\omega_{2n+1}^{a+b \mod (2n+1)},$$

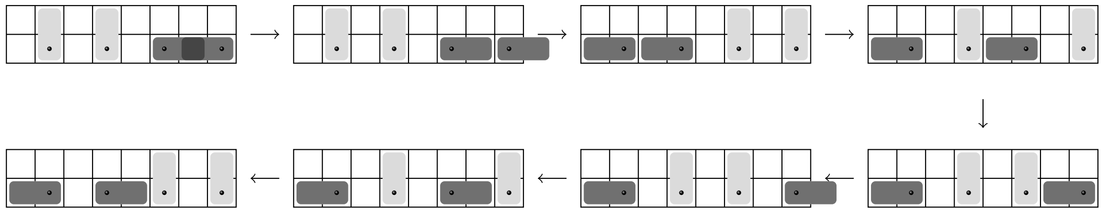
where $a,b\in\{\pm1,\pm2,\cdots,\pm n\}$ and $|a|\neq|b|$. Without loss of generality, we assume $|a|<|b|$. Then, the weight of P can be plotted in a Cartesian coordinate system. The graph below is an example with $n=4$:



It is easy to find a path in the above graph that includes exponents 1 to $2n$ of ω_{2n+1}:



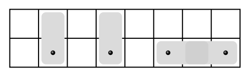
The pseudo-tilings in this path are



which have the sum of weights

$$\sum_{i=1}^{8} \omega_9^i = -1.$$

We need a pseudo-tiling from $X_{n,2}(0)$ to cancel all the weights of $X_{n,2}(2)$. An obvious choice is the following:



The weight of this pseudo-tiling is 1. Thus, by grouping pseudo-tilings in this way, the weight of each group is 0. Hence,

$$\mathrm{wt}(X_{n,2}) = \mathrm{wt}(\bigsqcup_{b=0}^{2} X_{n,2}(b)) = 0.$$

□

This set-grouping idea encountered great difficulties when we were trying to extend it to a bigger h. When $h \leq 2$, any pseudo-tiling with black pseudo-dominoes will not have a weight 1. But if $h > 2$, we can have pseudo-tilings with weights like the following:

$$\omega_{2n+1}^{1} \cdot \omega_{2n+1}^{2} \cdot \omega_{2n+1}^{-3} = \omega_{2n+1}^{0} = 1.$$

This is to say that not only $X_{n,h}(0)$ can contribute to a positive integer weight, but also some pseudo-tilings in $X_{n,h}(b)$, where $b > 2$. Finding positions such that black pseudo-dominoes can give a weight of 1 is a number theory question to find integers x_i such that

$$x_1 + x_2 + \cdots + x_k = 0 \mod 2n+1,$$

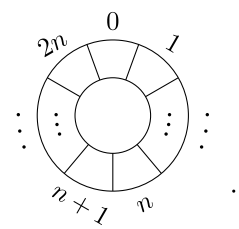
where $|x_1|, |x_2|, \cdots, |x_k| < n$ and are distinct. Hence, finding a universal set-grouping rule for $X_{n,h}$ is hard. Instead of directly describing the grouping of sets, we use a special kind of necklace tiling to tackle the problem in the next section.

4.2 Combinatorics Using Necklace Tiling

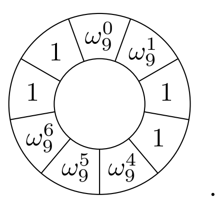
Theorem 4.2.1. *For arbitrary n,*

$$\mathrm{wt}(X_{n,n}) = 0.$$

Proof. We introduce a new tiling. We want to tile a necklace of length $2n+1$ using $2n+1$ dominoes. We number each cell in the necklace clockwise from the very top:

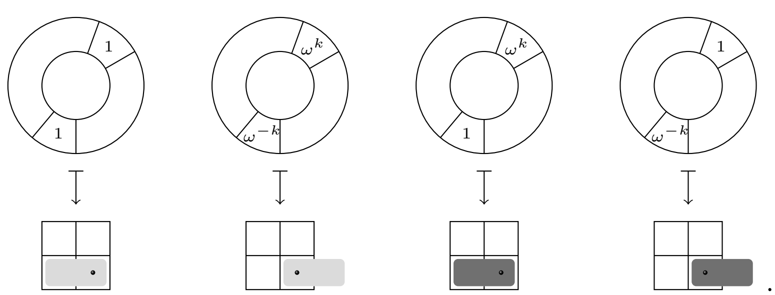


We can tile each cell by either a weight 1 domino or a weight $\omega^k := \omega^k_{2n+1}$ domino, where k is the position of the weighted domino. An example of a length 9 necklace tiling ϕ is

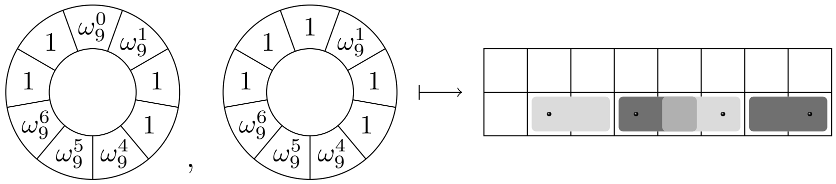


Note that for $k > n$, we can rewrite $\omega^k_{2n+1} = \omega^{-(k-n)}_{2n+1}$. We distinguish between ω^0_{2n+1} and 1 in the 0^{th} position, even though they have the same weight. We will explain this later when we describe the rotation of necklace tilings.

For $k \neq 0$, we can find a bijection between dominoes in $(2n+1)$-length necklace tilings and those in $2 \times 2n$ pseudo-tilings. Looking at the k^{th} and the $(n+k)^{th}$ position of the necklace tiling,



Note that the weight of dominoes in necklace tilings is equivalent to the weight of the corresponding dominoes in pseudo-tilings. Depending on the weight of the domino at the 0^{th} place, we have a two-to-one relation between $2n+1$ length necklace tilings and $2 \times 2n$ pseudo-tilings. Using the previous example of ϕ,



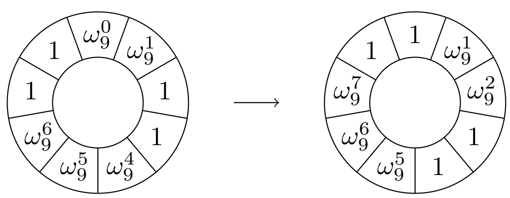
Note that the weight of the necklace tilings is the same as the weight of their corresponding pseudo-tiling.

Let a be the number of dominoes with weights of the form ω_{2n+1}^k, where $k \in \{0, 1, \cdots, 2n\}$. Let Φ_a be the set of all necklace tilings with a number of dominoes with roots of unity weights.

We can rotate a specific necklace tiling by multiplying the weight of it by ω_{2n+1}^a. For example, using the previous necklace tiling $\phi \in \Phi_5$ with $2n+1 = 9$, we multiply $\mathrm{wt}(\phi)$ by ω_9^5:

$$\begin{aligned}
\omega_9^5 \cdot \mathrm{wt}(\phi) &= \omega_9^5 \cdot \omega_9^0 \omega_9^1 \omega_9^4 \omega_9^5 \omega_9^6 \\
&= \omega_9 \omega_9^0 \cdot \omega_9 \omega_9^1 \cdot \omega_9 \omega_9^4 \cdot \omega_9 \omega_9^5 \cdot \omega_9 \omega_9^6 \\
&= \omega_9^1 \omega_9^2 \omega_9^5 \omega_9^6 \omega_9^7.
\end{aligned}$$

The rotation only goes once:



The resulting tiling is still an element of Φ_5. We distinguish between ω_{2n+1}^0 and 1 in the 0^{th} position because we want to make sure that rotation does not change a in a necklace tiling φ. We can conclude with the following equality:

$$\begin{aligned}
\omega_{2n+1}^a \sum_{\varphi \in \Phi_a} \mathrm{wt}(\varphi) &= \sum_{\varphi \in \Phi_a} \mathrm{wt}(\varphi) \\
\Longrightarrow \omega_{2n+1}^a \sum_{\varphi \in \Phi_a} \mathrm{wt}(\varphi) - \sum_{\varphi \in \Phi_a} \mathrm{wt}(\varphi) &= 0 \\
\Longrightarrow (\omega_{2n+1}^a - 1) \sum_{\varphi \in \Phi_a} \mathrm{wt}(\varphi) &= 0,
\end{aligned}$$

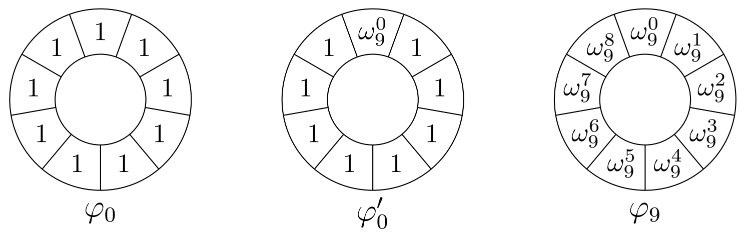
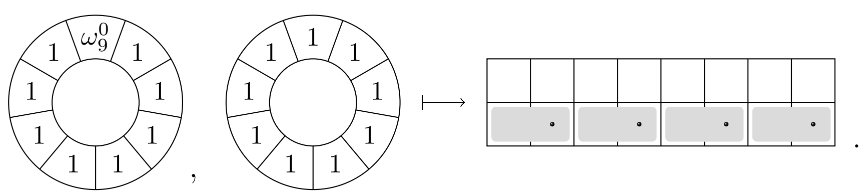


Figure 4.2: φ_0, φ_0' and φ_{2n+1}.

for any $\varphi \in \Phi_a$. If $\omega_{2n+1}^a - 1 \neq 0$, then $\sum_{\varphi \in \Phi_a} \mathrm{wt}(\varphi) = 0$; and $\omega_{2n+1}^a - 1 = 0$ only if $a = 2n+1$ or 0. If $a \neq 0$ or $2n+1$, then Φ_a has weight of 0.

Let φ_0 be the only element in Φ_0 and let φ_{2n+1} be the only element in Φ_{2n+1}. Let φ_0' be the counterpart of φ_0 that maps to the same pseudo-tiling. The following example gives φ_0, φ_0' and φ_{2n+1} when $2n+1 = 9$, see Figure 4.2. Note that $\varphi_0' \in \Phi_1$. Since $\mathrm{wt}(\varphi_0')$ $=\mathrm{wt}(\varphi_{2n+1}) = 1$, we can replace φ_0' in Φ_1 by φ_{2n+1} without changing the total weight of Φ_1. Now the two necklace tilings that make the weight of Φ_0 and Φ_{2n+1} not zero are φ_0 and φ_0'. These two necklace tilings map to the only proper $2 \times 2n$ tiling in $Y_{n,n}$:



All the other necklace tilings have weight of 0. Since there is a two-to-one relation between necklace tilings and pseudo-tilings,

$$\sum_{q \neq 0, 2n+1} \mathrm{wt}(\Phi_a) = 2 \cdot \mathrm{wt}(X_{n,n}) = 0.$$

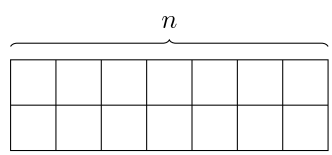
and thus $\mathrm{wt}(X_{n,n}) = 0$. □

Remark 4.2.2. The necklace tiling idea is inspired by a graduate thesis by Wahyuni [13] and the following algebra.

Let C be a $2 \times 2n$ checkerboard. By Example 3.3.2, $V_n(x)$ is the generating function for weights of pseudo-tilings of C with all horizontal pseudo-dominoes. Leting $x = 1$, it follows that $V_n(1)$ gives the number of pseudo-tilings of C whose pseudo-dominoes are all horizontal. Simple algebra gives:

$$\begin{aligned}
V_n(1) &= \prod_{k=1}^{n}(2+\omega_{2n+1}^{k}+\omega_{2n+1}^{-k}) \\
&= \prod_{k=1}^{n}(1+\omega_{2n+1}^{k}+\omega_{2n+1}^{-k}+\omega_{2n+1}^{k}\omega_{2n+1}^{-k}) \\
&= \prod_{k=1}^{n}(1+\omega_{2n+1}^{k})(\omega_{2n+1}^{k}\omega_{2n+1}^{-k}+\omega_{2n+1}^{-k}) \\
&= \prod_{k=1}^{n}(1+\omega_{2n+1}^{k})\prod_{k=0}^{n}(1+\omega_{2n+1}^{-k})/2.
\end{aligned}$$

We see the above product as ways to tile a $2 \times n$ checkerboard with 1×1 weighted dominoes:

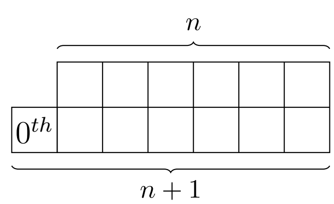


.

The squares on the top layer are tiled by either a weight 1 or a weight ω_{2n+1}^{k} domino, with k depends on the position of the tile. Similarly, squares on the bottom layer are tiled by either a weight 1 or a weight ω_{2n+1}^{-k} domino. We can further rewrite $V_n(1)$ as

$$V_n(1) = \prod_{k=1}^{n}(1+\omega_{2n+1}^{k})\prod_{k=1}^{n}(1+\omega_{2n+1}^{-k}) = \prod_{k=1}^{n}(1+\omega_{2n+1}^{k})\prod_{k=0}^{n}(1+\omega_{2n+1}^{-k})/2, \tag{4.1}$$

and we can attach a square to the left of the bottom layer, and that square is the 0^{th} cell:



.

Sticking the 0^{th} square to the 1^{st} square on the top layer, we get the necklace tiling.

Conclusion

In Chapter 3, we gave an algebraic proof that $\mathrm{wt}(X_{n,h}) = 0$ and in Chapter 4 we sketched a combinatorial proof of certain cases of h. We extended the set-grouping idea motivated by Chen to show the case for $h = 1$ and $h = 2$. For $h > 2$, it is difficult to find universal set-grouping rules. We switched to necklace tilings in order to have a cyclic group acting on a set of pseudo-tilings.

Some potential future work can focus on extending the proof of Theorem 4.2.1 to the case of arbitrary h. One can also consider extending the proof of Theorem 4.2.1 to the bigger checkerboard of $2m \times 2n$.

One may also explore Chebyshev polynomials of the third kind. The recursion for $V_n(x)$ implies a recursion for $G_n(h, x)$. These recursion imply interesting identities among roots of unity and weights of pseudo-tilings.

This thesis follows Benjamin and Chen's set-up to try to give a combinatorial proof of the $2m \times 2n$ case of Kasteleyn's tiling formula. One can also seek a way to extend their ideas to a $2m \times n$ checkerboard.

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